Symbolic Dynamics and Dynamical System Models

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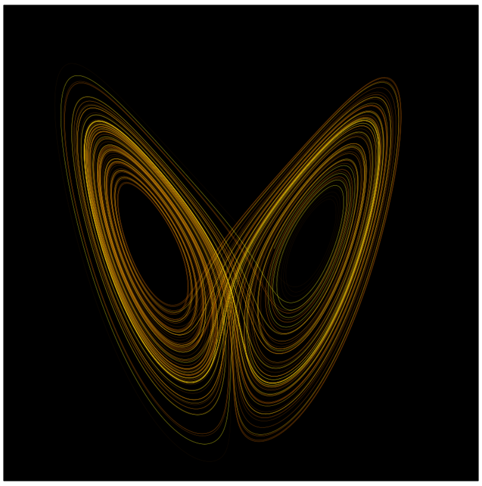
Articles

Edited and compiled by Bci2

Dynamical Systems and Symbolic Dynamics

Dynamical system

The **dynamical system** concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each spring in a lake.



The Lorenz attractor is an example of a non-linear dynamical system. Studying this system helped give rise to Chaos theory.

At any given time a dynamical system has a *state* given by a set of real numbers (a vector) which can be represented by a point in an appropriate *state space* (a geometrical manifold). Small changes in the state of the system correspond to small changes in the numbers. The *evolution rule* of the dynamical system is a fixed rule that describes what future states follow from the current state. The rule is deterministic: for a given time interval only one future state follows from the current state.

Overview

The concept of a dynamical system has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of dynamical systems is given implicitly by a relation that gives the state of the system only a short time into the future. (The relation is either a differential equation, difference equation or other time scale.) To determine the state for all future times requires iterating the relation many times—each advancing time a small step. The iteration procedure is referred to as *solving the system* or *integrating the system*. Once the system can be solved, given an initial point it is possible to determine all its future points, a collection known as a *trajectory* or *orbit*.

Before the advent of fast computing machines, solving a dynamical system required sophisticated mathematical techniques and could only be accomplished for a small class of dynamical systems. Numerical methods executed on computers have simplified the task of determining the orbits of a dynamical system.

For simple dynamical systems, knowing the trajectory is often sufficient, but most dynamical systems are too complicated to be understood in terms of individual trajectories. The difficulties arise because:

- The systems studied may only be known approximately—the parameters of the system may not be known precisely or terms may be missing from the equations. The approximations used bring into question the validity or relevance of numerical solutions. To address these questions several notions of stability have been introduced in the study of dynamical systems, such as Lyapunov stability or structural stability. The stability of the dynamical system implies that there is a class of models or initial conditions for which the trajectories would be equivalent. The operation for comparing orbits to establish their equivalence changes with the different notions of stability.
- The type of trajectory may be more important than one particular trajectory. Some trajectories may be periodic, whereas others may wander through many different states of the system. Applications often require enumerating these classes or maintaining the system within one class. Classifying all possible trajectories has led to the qualitative study of dynamical systems, that is, properties that do not change under coordinate changes. Linear

dynamical systems and systems that have two numbers describing a state are examples of dynamical systems where the possible classes of orbits are understood.
- The behavior of trajectories as a function of a parameter may be what is needed for an application. As a parameter is varied, the dynamical systems may have bifurcation points where the qualitative behavior of the dynamical system changes. For example, it may go from having only periodic motions to apparently erratic behavior, as in the transition to turbulence of a fluid.
- The trajectories of the system may appear erratic, as if random. In these cases it may be necessary to compute averages using one very long trajectory or many different trajectories. The averages are well defined for → ergodic systems and a more detailed understanding has been worked out for hyperbolic systems. Understanding the probabilistic aspects of dynamical systems has helped establish the foundations of statistical mechanics and of chaos.

It was in the work of Poincaré that these dynamical systems themes developed.

Basic definitions

A dynamical system is a manifold *M* called the phase (or state) space and a smooth evolution function Φ^t that for any element of $t \in T$, the time, maps a point of the phase space back into the phase space. The notion of smoothness changes with applications and the type of manifold. There are several choices for the set *T*. When *T* is taken to be the reals, the dynamical system is called a *flow*; and if *T* is restricted to the non-negative reals, then the dynamical system is a *semi-flow*. When *T* is taken to be the integers, it is a *cascade* or a *map*; and the restriction to the non-negative integers is a *semi-cascade*.

Examples

The evolution function Φ^t is often the solution of a *differential equation of motion*

$$\dot{x} = v(x)\,.$$

The equation gives the time derivative, represented by the dot, of a trajectory *x(t)* on the phase space starting at some point x_0. The *vector field v(x)* is a smooth function that at every point of the phase space *M* provides the velocity vector of the dynamical system at that point. (These vectors are not vectors in the phase space *M*, but in the tangent space TM_x of the point *x*.) Given a smooth Φ^t, an autonomous vector field can be derived from it.

There is no need for higher order derivatives in the equation, nor for time dependence in *v(x)* because these can be eliminated by considering systems of higher dimensions. Other types of differential equations can be used to define the evolution rule:

$$G(x, \dot{x}) = 0$$

is an example of an equation that arises from the modeling of mechanical systems with complicated constraints.

The differential equations determining the evolution function Φ^t are often ordinary differential equations: in this case the phase space *M* is a finite dimensional manifold. Many of the concepts in dynamical systems can be extended to infinite-dimensional manifolds—those that are locally Banach spaces—in which case the differential equations are partial differential equations. In the late 20th century the dynamical system perspective to partial differential equations started gaining popularity.

Further examples

- Logistic map
- Double pendulum
- Arnold's cat map
- Horseshoe map
- Baker's map is an example of a chaotic piecewise linear map
- Billiards and outer billiards
- Hénon map
- Lorenz system
- Circle map
- Rössler map
- List of chaotic maps
- Swinging Atwood's machine
- Quadratic map simulation system
- Bouncing ball simulation system

Linear dynamical systems

Linear dynamical systems can be solved in terms of simple functions and the behavior of all orbits classified. In a linear system the phase space is the N-dimensional Euclidean space, so any point in phase space can be represented by a vector with N numbers. The analysis of linear systems is possible because they satisfy a superposition principle: if *u(t)* and *w(t)* satisfy the differential equation for the vector field (but not necessarily the initial condition), then so will *u(t) + w(t)*.

Flows

For a flow, the vector field *Φ(x)* is a linear function of the position in the phase space, that is,

$$\phi(x) = Ax + b\,,$$

with A a matrix, b a vector of numbers and x the position vector. The solution to this system can be found by using the superposition principle (linearity). The case $b \neq 0$ with $A = 0$ is just a straight line in the direction of b:

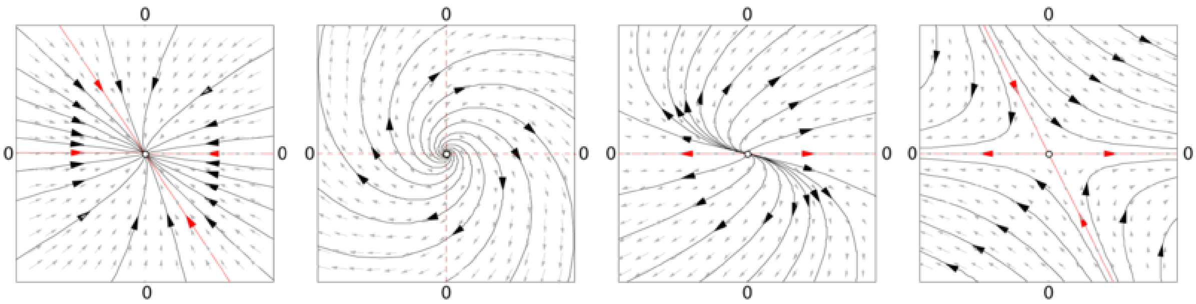
$$\Phi^t(x_1) = x_1 + bt\,.$$

When b is zero and $A \neq 0$ the origin is an equilibrium (or singular) point of the flow, that is, if $x_0 = 0$, then the orbit remains there. For other initial conditions, the equation of motion is given by the exponential of a matrix: for an initial point x_0,

$$\Phi^t(x_0) = e^{tA}x_0\,.$$

When $b = 0$, the eigenvalues of A determine the structure of the phase space. From the eigenvalues and the eigenvectors of A it is possible to determine if an initial point will converge or diverge to the equilibrium point at the origin.

The distance between two different initial conditions in the case $A \neq 0$ will change exponentially in most cases, either converging exponentially fast towards a point, or diverging exponentially fast. Linear systems display sensitive dependence on initial conditions in the case of divergence. For nonlinear systems this is one of the (necessary but not sufficient) conditions for chaotic behavior.



Linear vector fields and a few trajectories.

Maps

A discrete-time, affine dynamical system has the form

$$x_{n+1} = Ax_n + b\,,$$

with A a matrix and b a vector. As in the continuous case, the change of coordinates $x \to x + (1 - A)^{-1}b$ removes the term b from the equation. In the new coordinate system, the origin is a fixed point of the map and the solutions are of the linear system $A^n x_0$. The solutions for the map are no longer curves, but points that hop in the phase space. The orbits are organized in curves, or fibers, which are collections of points that map into themselves under the action of the map.

As in the continuous case, the eigenvalues and eigenvectors of A determine the structure of phase space. For example, if u_1 is an eigenvector of A, with a real eigenvalue smaller than one, then the straight lines given by the points along $\alpha\, u_1$, with $\alpha \in \mathbf{R}$, is an invariant curve of the map. Points in this straight line run into the fixed point.

There are also many other discrete dynamical systems.

Local dynamics

The qualitative properties of dynamical systems do not change under a smooth change of coordinates (this is sometimes taken as a definition of qualitative): a *singular point* of the vector field (a point where $v(x) = 0$) will remain a singular point under smooth transformations; a *periodic orbit* is a loop in phase space and smooth deformations of the phase space cannot alter it being a loop. It is in the neighborhood of singular points and periodic orbits that the structure of a phase space of a dynamical system can be well understood. In the qualitative study of dynamical systems, the approach is to show that there is a change of coordinates (usually unspecified, but computable) that makes the dynamical system as simple as possible.

Rectification

A flow in most small patches of the phase space can be made very simple. If y is a point where the vector field $v(y) \neq 0$, then there is a change of coordinates for a region around y where the vector field becomes a series of parallel vectors of the same magnitude. This is known as the rectification theorem.

The rectification theorem says that away from singular points the dynamics of a point in a small patch is a straight line. The patch can sometimes be enlarged by stitching several patches together, and when this works out in the whole phase space M the dynamical system is *integrable*. In most cases the patch cannot be extended to the entire phase space. There may be singular points in the vector field (where $v(x) = 0$); or the patches may become smaller and smaller as some point is approached. The more subtle reason is a global constraint, where the trajectory starts out in a patch, and after visiting a series of other patches comes back to the original one. If the next time the orbit loops around phase space in a different way, then it is impossible to rectify the vector field in the whole series of patches.

Near periodic orbits

In general, in the neighborhood of a periodic orbit the rectification theorem cannot be used. Poincaré developed an approach that transforms the analysis near a periodic orbit to the analysis of a map. Pick a point x_0 in the orbit γ and consider the points in phase space in that neighborhood that are perpendicular to $v(x_0)$. These points are a Poincaré section $S(\gamma, x_0)$, of the orbit. The flow now defines a map, the Poincaré map $F : S \to S$, for points starting in S and returning to S. Not all these points will take the same amount of time to come back, but the times will be close to the time it takes x_0.

The intersection of the periodic orbit with the Poincaré section is a fixed point of the Poincaré map F. By a translation, the point can be assumed to be at $x = 0$. The Taylor series of the map is $F(x) = J \cdot x + O(x^2)$, so a change of coordinates h can only be expected to simplify F to its linear part

$$h^{-1} \circ F \circ h(x) = J \cdot x\,.$$

This is known as the conjugation equation. Finding conditions for this equation to hold has been one of the major tasks of research in dynamical systems. Poincaré first approached it assuming all functions to be analytic and in the process discovered the non-resonant condition. If $\lambda_1,\ldots,\lambda_\nu$ are the eigenvalues of J they will be resonant if one eigenvalue is an integer linear combination of two or more of the others. As terms of the form $\lambda_i - \sum$ (multiples of other eigenvalues) occurs in the denominator of the terms for the function h, the non-resonant condition is also known as the small divisor problem.

Conjugation results

The results on the existence of a solution to the conjugation equation depend on the eigenvalues of J and the degree of smoothness required from h. As J does not need to have any special symmetries, its eigenvalues will typically be complex numbers. When the eigenvalues of J are not in the unit circle, the dynamics near the fixed point x_0 of F is called *hyperbolic* and when the eigenvalues are on the unit circle and complex, the dynamics is called *elliptic*.

In the hyperbolic case the Hartman-Grobman theorem gives the conditions for the existence of a continuous function that maps the neighborhood of the fixed point of the map to the linear map $J \cdot x$. The hyperbolic case is also *structurally stable*. Small changes in the vector field will only produce small changes in the Poincaré map and these small changes will reflect in small changes in the position of the eigenvalues of J in the complex plane, implying that the map is still hyperbolic.

The Kolmogorov-Arnold-Moser (KAM) theorem gives the behavior near an elliptic point.

Bifurcation theory

When the evolution map Φ^t (or the vector field it is derived from) depends on a parameter μ, the structure of the phase space will also depend on this parameter. Small changes may produce no qualitative changes in the phase space until a special value μ_0 is reached. At this point the phase space changes qualitatively and the dynamical system is said to have gone through a bifurcation.

Bifurcation theory considers a structure in phase space (typically a fixed point, a periodic orbit, or an invariant torus) and studies its behavior as a function of the parameter μ. At the bifurcation point the structure may change its stability, split into new structures, or merge with other structures. By using Taylor series approximations of the maps and an understanding of the differences that may be eliminated by a change of coordinates, it is possible to catalog the bifurcations of dynamical systems.

The bifurcations of a hyperbolic fixed point x_0 of a system family F_μ can be characterized by the eigenvalues of the first derivative of the system $DF_\mu(x_0)$ computed at the bifurcation point. For a map, the bifurcation will occur when there are eigenvalues of DF_μ on the unit circle. For a flow, it will occur when there are eigenvalues on the imaginary axis. For more information, see the main article on Bifurcation theory.

Some bifurcations can lead to very complicated structures in phase space. For example, the Ruelle-Takens scenario describes how a periodic orbit bifurcates into a torus and the torus into a strange attractor. In another example, Feigenbaum period-doubling describes how a stable periodic orbit goes through a series of period-doubling bifurcations.

Ergodic systems

In many dynamical systems it is possible to choose the coordinates of the system so that the volume (really a ν-dimensional volume) in phase space is invariant. This happens for mechanical systems derived from Newton's laws as long as the coordinates are the position and the momentum and the volume is measured in units of (position) × (momentum). The flow takes points of a subset *A* into the points Φ^t(A) and invariance of the phase space means that

$$\mathrm{vol}(A) = \mathrm{vol}(\Phi^t(A)) .$$

In the Hamiltonian formalism, given a coordinate it is possible to derive the appropriate (generalized) momentum such that the associated volume is preserved by the flow. The volume is said to be computed by the Liouville measure.

In a Hamiltonian system not all possible configurations of position and momentum can be reached from an initial condition. Because of energy conservation, only the states with the same energy as the initial condition are accessible. The states with the same energy form an energy shell Ω, a sub-manifold of the phase space. The volume of the energy shell, computed using the Liouville measure, is preserved under evolution.

For systems where the volume is preserved by the flow, Poincaré discovered the recurrence theorem: Assume the phase space has a finite Liouville volume and let *F* be a phase space volume-preserving map and *A* a subset of the phase space. Then almost every point of *A* returns to *A* infinitely often. The Poincaré recurrence theorem was used by Zermelo to object to Boltzmann's derivation of the increase in entropy in a dynamical system of colliding atoms.

One of the questions raised by Boltzmann's work was the possible equality between time averages and space averages, what he called the ergodic hypothesis. The hypothesis states that the length of time a typical trajectory spends in a region *A* is vol(*A*)/vol(Ω).

The ergodic hypothesis turned out not to be the essential property needed for the development of statistical mechanics and a series of other ergodic-like properties were introduced to capture the relevant aspects of physical systems. Koopman approached the study of ergodic systems by the use of functional analysis. An observable *a* is a function that to each point of the phase space associates a number (say instantaneous pressure, or average height). The value of an observable can be computed at another time by using the evolution function φ^t. This introduces an operator U^t, the transfer operator,

$$(U^t a)(x) = a(\Phi^{-t}(x)) .$$

By studying the spectral properties of the linear operator *U* it becomes possible to classify the ergodic properties of Φ^t. In using the Koopman approach of considering the action of the flow on an observable function, the finite-dimensional nonlinear problem involving Φ^t gets mapped into an infinite-dimensional linear problem involving *U*.

The Liouville measure restricted to the energy surface Ω is the basis for the averages computed in equilibrium statistical mechanics. An average in time along a trajectory is equivalent to an average in space computed with the Boltzmann factor exp(−β*H*). This idea has been generalized by Sinai, Bowen, and Ruelle (SRB) to a larger class of dynamical systems that includes dissipative systems. SRB measures replace the Boltzmann factor and they are defined on attractors of chaotic systems.

Nonlinear dynamical systems and chaos

Simple nonlinear dynamical systems and even piecewise linear systems can exhibit a completely unpredictable behavior, which might seem to be random. (Remember that we are speaking of completely deterministic systems!). This seemingly unpredictable behavior has been called *chaos*. Hyperbolic systems are precisely defined dynamical systems that exhibit the properties ascribed to chaotic systems. In hyperbolic systems the tangent space perpendicular to a trajectory can be well separated into two parts: one with the points that converge towards the orbit (the *stable manifold*) and another of the points that diverge from the orbit (the *unstable manifold*).

This branch of mathematics deals with the long-term qualitative behavior of dynamical systems. Here, the focus is not on finding precise solutions to the equations defining the dynamical system (which is often hopeless), but rather to answer questions like "Will the system settle down to a steady state in the long term, and if so, what are the possible attractors?" or "Does the long-term behavior of the system depend on its initial condition?"

Note that the chaotic behavior of complicated systems is not the issue. Meteorology has been known for years to involve complicated—even chaotic—behavior. Chaos theory has been so surprising because chaos can be found within almost trivial systems. The logistic map is only a second-degree polynomial; the horseshoe map is piecewise linear.

Geometrical definition

A dynamical system is the tuple $\langle \mathcal{M}, f, \mathcal{T} \rangle$, with $\mathcal{M}$ a manifold (locally a Banach space or Euclidean space), $\mathcal{T}$ the domain for time (non-negative reals, the integers, ...) and f an evolution rule t→f^t (with $t \in \mathcal{T}$) such that f^t is a diffeomorphism of the manifold to itself. So, f is a mapping of the time-domain $\mathcal{T}$ into the space of diffeomorphisms of the manifold to itself. In other terms, f(t) is a diffeomorphism, for every time t in the domain $\mathcal{T}$.

Measure theoretical definition

See main article → Measure-preserving dynamical system.

A dynamical system may be defined formally, as a measure-preserving transformation of a sigma-algebra, the quadruplet (X, Σ, μ, τ). Here, *X* is a set, and Σ is a sigma-algebra on *X*, so that the pair (X, Σ) is a measurable space. μ is a finite measure on the sigma-algebra, so that the triplet (X, Σ, μ) is a probability space. A map $\tau : X \to X$ is said to be Σ-measurable if and only if, for every $\sigma \in \Sigma$, one has $\tau^{-1}\sigma \in \Sigma$. A map τ is said to **preserve the measure** if and only if, for every $\sigma \in \Sigma$, one has $\mu(\tau^{-1}\sigma) = \mu(\sigma)$. Combining the above, a map τ is said to be a **measure-preserving transformation of *X*** , if it is a map from *X* to itself, it is Σ-measurable, and is measure-preserving. The quadruple (X, Σ, μ, τ), for such a τ, is then defined to be a **dynamical system**.

The map τ embodies the time evolution of the dynamical system. Thus, for discrete dynamical systems the iterates $\tau^n = \tau \circ \tau \circ \ldots \circ \tau$ for integer *n* are studied. For continuous dynamical systems, the map τ is understood to be finite time evolution map and the construction is more complicated.

Examples of dynamical systems

Wikipedia links

- Arnold's cat map
- Baker's map is an example of a chaotic piecewise linear map
- Circle map
- Double pendulum
- Billiards and Outer Billiards
- Henon map
- Horseshoe map
- Irrational rotation
- List of chaotic maps
- Logistic map
- Lorenz system
- Rossler map

External links

- Bouncing Ball [1]
- Mechanical Strings [2]
- Journal of Advanced Research in Dynamical and Control Systems [3]
- Swinging Atwood's Machine (SAM) [4]
- Interactive applet for the Standard and Henon Maps [5] by A. Luhn

See also

- Behavioral modeling
- → Dynamical systems theory
- List of dynamical system topics
- Oscillation
- People in systems and control
- Sarkovskii's theorem
- System dynamics
- Systems theory

Further reading

Works providing a broad coverage:

- Ralph Abraham and Jerrold E. Marsden (1978). *Foundations of mechanics*. Benjamin-Cummings. ISBN 0-8053-0102-X. (available as a reprint: ISBN 0-201-40840-6)
- *Encyclopaedia of Mathematical Sciences* (ISSN 0938-0396) has a sub-series on dynamical systems [6] with reviews of current research.
- Anatole Katok and Boris Hasselblatt (1996). *Introduction to the modern theory of dynamical systems*. Cambridge. ISBN 0-521-57557-5.
- Christian Bonatti, Lorenzo J. Díaz, Marcelo Viana (2005). *Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective*. Springer. ISBN 3-540-22066-6.
- Diederich Hinrichsen and Anthony J. Pritchard (2005). *Mathematical Systems Theory I - Modelling, State Space Analysis, Stability and Robustness*. Springer Verlag. ISBN 978-3-540-44125-0.

Introductory texts with a unique perspective:

- V. I. Arnold (1982). *Mathematical methods of classical mechanics*. Springer-Verlag. ISBN 0-387-96890-3.
- Jacob Palis and Wellington de Melo (1982). *Geometric theory of dynamical systems: an introduction*. Springer-Verlag. ISBN 0-387-90668-1.
- David Ruelle (1989). *Elements of Differentiable Dynamics and Bifurcation Theory*. Academic Press. ISBN 0-12-601710-7.
- Tim Bedford, Michael Keane and Caroline Series, *eds.* (1991). *Ergodic theory, symbolic dynamics and hyperbolic spaces*. Oxford University Press. ISBN 0-19-853390-X.
- Ralph H. Abraham and Christopher D. Shaw (1992). *Dynamics—the geometry of behavior, 2nd edition*. Addison-Wesley. ISBN 0-201-56716-4.

Textbooks

- Steven H. Strogatz (1994). *Nonlinear dynamics and chaos: with applications to physics, biology chemistry and engineering*. Addison Wesley. ISBN 0-201-54344-3.
- Kathleen T. Alligood, Tim D. Sauer and James A. Yorke (2000). *Chaos. An introduction to dynamical systems*. Springer Verlag. ISBN 0-387-94677-2.
- Morris W. Hirsch, Stephen Smale and Robert Devaney (2003). *Differential Equations, dynamical systems, and an introduction to chaos*. Academic Press. ISBN 0-12-349703-5.

Popularizations:

- Florin Diacu and Philip Holmes (1996). *Celestial Encounters*. Princeton. ISBN 0-691-02743-9.
- James Gleick (1988). *Chaos: Making a New Science*. Penguin. ISBN 0-14-009250-1.
- Ivar Ekeland (1990). *Mathematics and the Unexpected (Paperback)*. University Of Chicago Press. ISBN 0-226-19990-8.
- Ian Stewart (1997). *Does God Play Dice? The New Mathematics of Chaos*. Penguin. ISBN 0140256024.

External links

- A collection of dynamic and non-linear system models and demo applets [7] (in Monash University's Virtual Lab)
- Arxiv preprint server [8] has daily submissions of (non-refereed) manuscripts in dynamical systems.
- DSWeb [9] provides up-to-date information on dynamical systems and its applications.
- Encyclopedia of dynamical systems [10] A part of Scholarpedia — peer reviewed and written by invited experts.
- Nonlinear Dynamics [11]. Models of bifurcation and chaos by Elmer G. Wiens
- Oliver Knill [12] has a series of examples of dynamical systems with explanations and interactive controls.
- Sci.Nonlinear FAQ 2.0 (Sept 2003) [13] provides definitions, explanations and resources related to nonlinear science

Online books or lecture notes:

- Geometrical theory of dynamical systems [14]. Nils Berglund's lecture notes for a course at ETH at the advanced undergraduate level.
- Dynamical systems [15]. George D. Birkhoff's 1927 book already takes a modern approach to dynamical systems.
- Chaos: classical and quantum [16]. An introduction to dynamical systems from the periodic orbit point of view.
- Modeling Dynamic Systems [17]. An introduction to the development of mathematical models of dynamic systems.
- Learning Dynamical Systems [18]. Tutorial on learning dynamical systems.
- Ordinary Differential Equations and Dynamical Systems [19]. Lecture notes by Gerald Teschl

Research groups:

- Dynamical Systems Group Groningen [20], IWI, University of Groningen.
- Chaos @ UMD [21]. Concentrates on the applications of dynamical systems.

- Dynamical Systems [22], SUNY Stony Brook. Lists of conferences, researchers, and some open problems.
- Center for Dynamics and Geometry [23], Penn State.
- Control and Dynamical Systems [24], Caltech.
- Laboratory of Nonlinear Systems [25], Ecole Polytechnique Fédérale de Lausanne (EPFL).
- Center for Dynamical Systems [26], University of Bremen
- Systems Analysis, Modelling and Prediction Group [27], University of Oxford
- Non-Linear Dynamics Group [28], Instituto Superior Técnico, Technical University of Lisbon
- Dynamical Systems [29], IMPA, Instituto Nacional de Matemática Pura e Aplicada.
- Nonlinear Dynamics Workgroup [30], Institute of Computer Science, Czech Academy of Sciences.

Simulation software based on Dynamical Systems approach:

- FyDiK [31]

References

[1] http://www.drchaos.net/drchaos/bb.html
[2] http://www.drchaos.net/drchaos/string_web_page/index.html
[3] http://www.i-asr.org/dynamic.html
[4] http://www.drchaos.net/drchaos/Sam/sam.html
[5] http://complexity.xozzox.de/nonlinmappings.html
[6] http://en.wikipedia.org/wiki/User:XaosBits/EMP
[7] http://vlab.infotech.monash.edu.au/simulations/non-linear/
[8] http://www.arxiv.org/list/math.DS/recent
[9] http://www.dynamicalsystems.org/
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[11] http://www.egwald.ca/nonlineardynamics/index.php
[12] http://www.dynamical-systems.org
[13] http://amath.colorado.edu/faculty/jdm/faq-Contents.html
[14] http://arxiv.org/pdf/math.HO/0111177
[15] http://www.ams.org/online_bks/coll9/
[16] http://chaosbook.org
[17] http://www.embedded.com/2000/0008/0008feat2.htm
[18] http://www.cs.brown.edu/research/ai/dynamics/tutorial/home.html
[19] http://www.mat.univie.ac.at/~gerald/ftp/book-ode/
[20] http://www.math.rug.nl/~broer/
[21] http://www-chaos.umd.edu/
[22] http://www.math.sunysb.edu/dynamics/
[23] http://www.math.psu.edu/dynsys/
[24] http://www.cds.caltech.edu/
[25] http://lanoswww.epfl.ch/
[26] http://www.math.uni-bremen.de/ids.html/
[27] http://www.eng.ox.ac.uk/samp/
[28] http://sd.ist.utl.pt/
[29] http://www.impa.br/
[30] http://ndw.cs.cas.cz/
[31] http://fydik.kitnarf.cz/

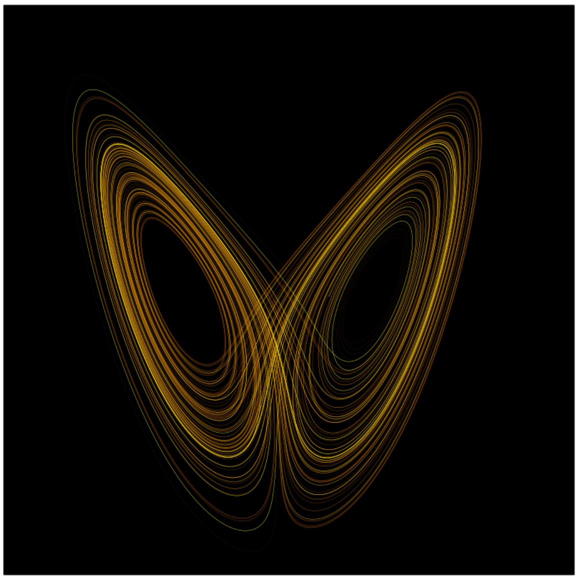
Dynamical systems theory

Dynamical systems theory is an area of applied mathematics used to describe the behavior of complex → dynamical systems, usually by employing differential equations or difference equations. When differential equations are employed, the theory is called *continuous dynamical systems*. When difference equations are employed, the theory is called *discrete dynamical systems*. When the time variable runs over a set which is discrete over some intervals and continuous over other intervals or is any arbitrary time-set such as a cantor set then one gets dynamic equations on time scales. Some situations may also be modelled by mixed operators such as differential-difference equations.

This theory deals with the long-term qualitative behavior of → dynamical systems, and the studies of the solutions to the equations of motion of systems that are primarily mechanical in nature; although this includes both planetary orbits as well as the behaviour of electronic circuits and the solutions to partial differential equations that arise in biology. Much of modern research is focused on the study of chaotic systems.

This field of study is also called just *Dynamical systems*, *Systems theory* or longer as *Mathematical Dynamical Systems Theory* and the *Mathematical theory of dynamical systems*.

Overview


The Lorenz attractor is an example of a non-linear dynamical system. Studying this system helped give rise to Chaos theory.

Dynamical systems theory and chaos theory deal with the long-term qualitative behavior of → dynamical systems. Here, the focus is not on finding precise solutions to the equations defining the dynamical system (which is often hopeless), but rather to answer questions like "Will the system settle down to a steady state in the long term, and if so, what are the possible steady states?", or "Does the long-term behavior of the system depend on its initial condition?"

An important goal is to describe the fixed points, or steady states of a given dynamical system; these are values of the variable which won't change over time. Some of these fixed points are *attractive*, meaning that if the system starts out in a nearby state, it will converge towards the fixed point.

Similarly, one is interested in *periodic points*, states of the system which repeat themselves after several timesteps. Periodic points can also be attractive. Sarkovskii's theorem is an interesting statement about the number of periodic points of a one-dimensional discrete dynamical system.

Even simple nonlinear dynamical systems often exhibit almost random, completely unpredictable behavior that has been called *chaos*. The branch of dynamical systems which deals with the clean definition and investigation of chaos is called chaos theory.

History

The concept of dynamical systems theory has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of dynamical systems is given implicitly by a relation that gives the state of the system only a short time into the future.

Before the advent of fast computing machines, solving a dynamical system required sophisticated mathematical techniques and could only be accomplished for a small class of dynamical systems.

Some excellent presentations of mathematical dynamic system theory include Beltrami (1987), Luenberger (1979), Padula and Arbib (1974), and Strogatz (1994).[1]

Concepts

Dynamical systems

The → dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each spring in a lake.

A dynamical system has a *state* determined by a collection of real numbers, or more generally by a set of points in an appropriate *state space*. Small changes in the state of the system correspond to small changes in the numbers. The numbers are also the coordinates of a geometrical space—a manifold. The *evolution rule* of the dynamical system is a fixed rule that describes what future states follow from the current state. The rule is deterministic: for a given time interval only one future state follows from the current state.

Dynamicism

Dynamicism, also termed the *dynamic hypothesis* or the *dynamic hypothesis in cognitive science* or *dynamic cognition*, is a new approach in cognitive science exemplified by the work of philosopher Tim van Gelder. It argues that differential equations are more suited to modelling cognition than more traditional computer models.

Nonlinear system

In mathematics, a nonlinear system is a system which is not linear, i.e. a system which does not satisfy the superposition principle. Less technically, a nonlinear system is any problem where the variable(s) to be solved for cannot be written as a linear sum of independent components. A nonhomogenous system, which is linear apart from the presence of a function of the independent variables, is nonlinear according to a strict definition, but such systems are usually studied alongside linear systems, because they can be transformed to a linear system as long as a particular solution is known.

Related fields

Arithmetic dynamics

Arithmetic dynamics is a field that emerged in the 1990s that amalgamates two areas of mathematics, dynamical systems and number theory. Classically, discrete dynamics refers to the study of the iteration of self-maps of the complex plane or real line. Arithmetic dynamics is the study of the number-theoretic properties of integer, rational, *p*-adic, and/or algebraic points under repeated application of a polynomial or rational function.

Chaos theory

Chaos theory describes the behavior of certain dynamical systems – that is, systems whose state evolves with time – that may exhibit dynamics that are highly sensitive to initial conditions (popularly referred to as the butterfly effect). As a result of this sensitivity, which manifests itself as an exponential growth of perturbations in the initial conditions, the behavior of chaotic systems appears to be random. This happens even though these systems are deterministic, meaning that their future dynamics are fully defined by their initial conditions, with no random elements involved. This behavior is known as deterministic chaos, or simply *chaos*.

Complex systems

Complex systems is a scientific field, which studies the common properties of systems considered complex in nature, society and science. It is also called *complex systems theory*, *complexity science*, *study of complex systems* and/or *sciences of complexity*. The key problems of such systems are difficulties with their formal modeling and simulation. From such perspective, in different research contexts complex systems are defined on the base of their different attributes.

The study of complex systems is bringing new vitality to many areas of science where a more typical reductionist strategy has fallen short. *Complex systems* is therefore often used as a broad term encompassing a research approach to problems in many diverse disciplines including neurosciences, social sciences, meteorology, chemistry, physics, computer science, psychology, artificial life, evolutionary computation, economics, earthquake prediction, molecular biology and inquiries into the nature of living cells themselves.

Control theory

Control theory is an interdisciplinary branch of engineering and mathematics, that deals with influencing the behavior of → dynamical systems.

Ergodic theory

→ Ergodic theory is a branch of mathematics that studies → dynamical systems with an invariant measure and related problems. Its initial development was motivated by problems of statistical physics.

Functional analysis

Functional analysis is the branch of mathematics, and specifically of analysis, concerned with the study of vector spaces and operators acting upon them. It has its historical roots in the study of functional spaces, in particular transformations of functions, such as the Fourier transform, as well as in the study of differential and integral equations. This usage of the word *functional* goes back to the calculus of variations, implying a function whose argument is a function. Its use in general has been attributed to mathematician and physicist Vito Volterra and its founding is largely attributed to mathematician Stefan Banach.

Graph dynamical systems

The concept of → graph dynamical systems (GDS) can be used to capture a wide range of processes taking place on graphs or networks. A major theme in the mathematical and computational analysis of GDS is to relate their structural properties (e.g. the network connectivity) and the global dynamics that result.

Projected dynamical systems

Projected dynamical systems is a mathematical theory investigating the behaviour of → dynamical systems where solutions are restricted to a constraint set. The discipline shares connections to and applications with both the static world of optimization and equilibrium problems and the dynamical world of ordinary differential equations. A projected dynamical system is given by the flow to the projected differential equation.

Symbolic dynamics

→ Symbolic dynamics is the practice of modelling a topological or smooth → dynamical system by a discrete space consisting of infinite sequences of abstract symbols, each of which corresponds to a state of the system, with the dynamics (evolution) given by the → shift operator.

System dynamics

System dynamics is an approach to understanding the behaviour of complex systems over time. It deals with internal feedback loops and time delays that affect the behaviour of the entire system.[2] What makes using system dynamics different from other approaches to studying complex systems is the use of feedback loops and stocks and flows. These elements help describe how even seemingly simple systems display baffling nonlinearity.

Topological dynamics

→ Topological dynamics is a branch of the theory of dynamical systems in which qualitative, asymptotic properties of dynamical systems are studied from the viewpoint of general topology.

Applications

In biomechanics

In sports biomechanics, dynamical systems theory has emerged in the movement sciences as a viable framework for modeling athletic performance. From a dynamical systems perspective, the human movement system is a highly intricate network of co-dependent sub-systems (e.g. respiratory, circulatory, nervous, skeletomuscular, perceptual) that are composed of a large number of interacting components (e.g. blood cells, oxygen molecules, muscle tissue, metabolic enzymes, connective tissue and bone). In dynamical systems theory, movement patterns emerge through generic processes of self-organization found in physical and biological systems.[3]

In cognitive science

Dynamical system theory has been applied in the field of neuroscience and cognitive development. It is the belief that cognitive development is best represented by physical theories rather than theories based on syntax and AI. It also believes that differential equations are the most appropriate tool for modeling human behavior. These equations are interpreted to represent an agent's cognitive trajectory through state space. In other words, dynamicists argue that psychology should be (or is) the description (via differential equations) of the cognitions and behaviors of an agent under certain environmental and internal pressures. The language of chaos theory is also frequently adopted.

In it, the learner's mind reaches a state of disequilibrium where old patterns have broken down. This is the phase transition of cognitive development. Self organization (the spontaneous creation of coherent forms) sets in as activity levels link to each other. Newly formed macroscopic and microscopic structures support each other, speeding up the process. These links form the structure of a new state of order in the mind through a process called *scalloping* (the repeated building up and collapsing of complex performance.) This new, novel state is progressive, discrete, idiosyncratic and unpredictable. [4]

Dynamic systems theory has recently been used to explain a long-unanswered problem in child development referred to as the A-not-B error.[5]

See also

Related subjects

- List of dynamical system topics
- Baker's map
- Dynamical system (definition)
- Embodied Embedded Cognition
- Gingerbreadman map
- Halo orbit
- List of types of systems theory
- Oscillation
- Postcognitivism
- Recurrent neural network
- Combinatorics and dynamical systems
- Synergetics

Related scientists

- People in systems and control
- Dmitri Anosov
- Vladimir Arnold
- Nikolay Bogolyubov
- Andrey Kolmogorov
- Nikolay Krylov
- Jürgen Moser
- Yakov G. Sinai
- Stephen Smale
- Hillel Furstenberg

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External links

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- Definition of dynamical system [7] in MathWorld.
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Symbolic dynamics

In mathematics, **symbolic dynamics** is the practice of modelling a topological or smooth → dynamical system by a discrete space consisting of infinite sequences of abstract symbols, each of which corresponds to a state of the system, with the dynamics (evolution) given by the → shift operator.

History

The idea goes back to → Jacques Hadamard's 1898 paper on the geodesics on surfaces of negative curvature. It was applied by → Marston Morse in 1921 to the construction of a nonperiodic recurrent geodesic. Related work was done by → Emil Artin in 1924 (for the system now called Artin billiard), P. J. Myrberg, → Paul Koebe, Jakob Nielsen, → G. A. Hedlund.

The first formal treatment was developed by Morse and Hedlund in their 1938 paper. → George Birkhoff, Norman Levinson and M. L. Cartwright–J. E. Littlewood have applied similar methods to qualitative analysis of nonautonomous second order differential equations.

→ Claude Shannon used symbolic sequences and shifts of finite type in his 1948 paper *A mathematical theory of communication* that gave birth to information theory.

The theory was further advanced in the 1960s and 1970s, notably, in the works of → Steve Smale and his school, and of → Yakov Sinai and the Soviet school of → ergodic theory. A spectacular application of the methods of symbolic dynamics is → Sharkovskii's theorem about → periodic orbits of a continuous map of an interval into itself (1964).

Applications

Symbolic dynamics originated as a method to study general dynamical systems; now its techniques and ideas have found significant applications in → data storage and → transmission, linear algebra, the motions of the planets and many other areas. The distinct feature in symbolic dynamics is that time is measured in *discrete* intervals. So at each time interval the system is in a particular *state*. Each state is associated with a symbol and the evolution of the system is described by an infinite sequence of symbols — represented effectively as strings. If the system states are not inherently discrete, then the state vector must be discretized, so as to get a coarse-grained description of the system.

See also

- → Measure-preserving dynamical system
- → Shift space
- Shift of finite type
- → Markov partition

Further reading

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Basic Concepts in Symbolic Dynamics

Sequential dynamical system

Sequential dynamical systems (SDSs) are a class of → graph dynamical systems. They are discrete dynamical systems which generalize many aspects of for example classical cellular automata, and they provide a framework for studying asynchronous processes over graphs. The analysis of SDSs uses techniques from combinatorics, abstract algebra, graph theory, → dynamical systems and probability theory.

Definition

An SDS is constructed from the following components:

- A finite *graph* Y with vertex set v[Y] = {1,2, ... , n}. Depending on the context the graph can be directed or undirected.
- A state x_v for each vertex i of Y taken from a finite set K. The *system state* is the n-tuple $x = (x_1, x_2, \ldots , x_n)$, and $x[i]$ is the tuple consisting of the states associated to the vertices in the 1-neighborhood of i in Y (in some fixed order).
- A *vertex function* f_i for each vertex i. The vertex function maps the state of vertex i at time t to the vertex state at time t + 1 based on the states associated to the 1-neighborhood of i in Y.
- A word $w = (w_1, w_2, \ldots , w_m)$ over $v[Y]$.

It is convenient to introduce the Y-local maps F_i constructed from the vertex functions by

$$F_i(x) = (x_1, x_2, \ldots, x_{i-1}, f_i(x[i]), x_{i+1}, \ldots, x_n) \ .$$

The word w specifies the sequence in which the Y-local maps are composed to derive the sequential dynamical system map F: $K^n \to K^n$ as

$$[F_Y, w] = F_{w(m)} \circ F_{w(m-1)} \circ \cdots \circ F_{w(2)} \circ F_{w(1)} \ .$$

If the update sequence is a permutation one frequently speaks of a *permutation SDS* to emphasize this point. The *phase space* associated to a sequential dynamical system with map F: $K^n \to K^n$ is the finite directed graph with vertex set K^n and directed edges (x, $F(x)$). The structure of the phase space is governed by the properties of the graph Y, the vertex functions $(f_i)_i$, and the update sequence w. A large part of SDS research seeks to infer phase space properties based on the structure of the system constituents.

Example

Consider the case where Y is the graph with vertex set {1,2,3} and undirected edges {1,2}, {1,3} and {2,3} (a triangle or 3-circle) with vertex states from K = {0,1}. For vertex functions use the symmetric, boolean function nor : $K \to K^3$ defined by nor(x,y,z) = (1+x)(1+y)(1+z) with boolean arithmetic. Thus, the only case in which the function nor returns the value 1 is when all the arguments are 0. Pick w = (1,2,3) as update sequence. Starting from the initial system state (0,0,0) at time t = 0 one computes the state of vertex 1 at time t=1 as nor(0,0,0) = 1. The state of vertex 2 at time t=1 is nor(1,0,0) = 0. Note that the state of vertex 1 at time t=1 is used immediately. Next one obtains the state of vertex 3 at time t=1 as nor(1,0,0) = 0. This completes the update sequence, and one concludes that the Nor-SDS map sends the system state (0,0,0) to (1,0,0). The system state (1,0,0) is in turned mapped to (0,1,0) by an application of the SDS map.

See also

- → Graph dynamical system
- Boolean network
- Gene regulatory network
- → Dynamic Bayesian network
- Petri net

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Automata theory

In theoretical computer science, **automata theory** is the study of abstract machines and problems which they are able to solve. Automata theory is closely related to formal language theory as the automata are often classified by the class of formal languages they are able to recognize.

An automaton is a mathematical model for a finite state machine (FSM). A FSM is a machine that, given an input of symbols, "**jumps**", or *transitions*, through a series of states according to a transition function (which can be expressed as a table). In the common "Mealy" variety of FSMs, this transition function tells the automaton which state to go to next given a current state and a current symbol.

The input is *read* symbol by symbol, until it is consumed completely (similar to a tape with a word written on it, which is read by a reading head of the automaton; the head moves forward over the tape, reading one symbol at a time). Once the input is depleted, the automaton is said to have *stopped*.

Depending on the state in which the automaton stops, it's said that the automaton either *accepts* or *rejects* the input. If it landed in an *accept state*, then the automaton *accepts* the word. If, on the other hand, it lands on a *reject state*, the word is *rejected*. The set of all the words accepted by an automaton is called the *language accepted by the automaton*.

Note, however, that, in general, an automaton need not have a finite number of states, or even a countable number of states. Thus, for example, the quantum finite automaton has an uncountable infinity of states, as the set of all possible states is the set of all points in complex projective space. Thus, quantum finite automata, as well as finite state machines, are special cases of a more general idea, that of a topological automaton, where the set of states is a topological space, and the state transition functions are taken from the set of all possible functions on the space. Topological automata are often called M-automata, and are simply the augmentation of a semiautomaton with a set of accept states, where set intersection determines whether the initial state is accepted or rejected.

In general, an automaton need not strictly accept or reject an input; it may accept it with some probability between zero and one. Again this is illustrated by the quantum finite automaton, which only accepts input with some probability. This idea is again a special case of a more general notion, the geometric automaton or *metric automaton*, where the set of states is a metric space, and a language is accepted by the automaton if the distance between the

initial point, and the set of accept states is sufficiently small with respect to the metric.

Automata play a major role in compiler design and parsing.

Vocabulary

The basic concepts of *symbols*, *words*, *alphabets* and *strings* are common to most descriptions of automata. These are:

Symbol

An arbitrary datum which has some meaning to or effect on the machine. Symbols are sometimes just called "letters" or "atoms"[1] .

Word

A finite string formed by the concatenation of a number of symbols.

Alphabet

A finite set of symbols. An alphabet is frequently denoted by Σ, which is the set of letters in an alphabet.

Language

A set of words, formed by symbols in a given alphabet. May or may not be infinite.

Kleene closure

A language may be thought of as a subset of all possible words. The set of all possible words may, in turn, be thought of as the set of all possible concatenations of strings. Formally, this set of all possible strings is called a free monoid. It is denoted as Σ^* , and the superscript * is called the Kleene star.

Formal description

An **automaton** is represented by the 5-tuple $\langle Q, \Sigma, \delta, q_0, F \rangle$, where:

- Q is a set of *states*.
- ∑ is a finite set of *symbols*, that we will call the *alphabet* of the language the automaton accepts.
- δ is the **transition function**, that is

  $$\delta : Q \times \Sigma \to Q.$$

  (For non-deterministic automata, the empty string is an allowed input).
- q_0 is the *start state*, that is, the state in which the automaton *is* when no input has been processed yet, where $q_0 \in$ Q.
- F is a set of states of Q (i.e. F⊆Q) called **accept states**.

Given an input letter $a \in \Sigma$, one may write the transition function as $\delta_a : Q \to Q$, using the simple trick of currying, that is, writing $\delta(q, a) = \delta_a(q)$ for all $q \in Q$. This way, the transition function can be seen in simpler terms: it's just something that "acts" on a state in Q, yielding another state. One may then consider the result of function composition repeatedly applied to the various functions δ_a , δ_b , and so on. Repeated function composition forms a monoid. For the transition functions, this monoid is known as the transition monoid, or sometimes the *transformation semigroup*.

Given a pair of letters $a, b \in \Sigma$, one may define a new function $\widehat{\delta}$, by insisting that $\widehat{\delta}_{ab} = \delta_a \circ \delta_b$, where $\circ$ denotes function composition. Clearly, this process can be recursively continued, and so one has a recursive definition of a function $\widehat{\delta}_w$ that is defined for all words $w \in \Sigma^*$, so that one has a map

$$\widehat{\delta} : Q \times \Sigma^\star \to Q.$$

The construction can also be reversed: given a $\widehat{\delta}$, one can reconstruct a δ , and so the two descriptions are equivalent.

The triple $\langle Q, \Sigma, \delta \rangle$ is known as a semiautomaton. Semiautomata underlay automata, in that they are just automata where one has ignored the starting state and the set of accept states. The additional notions of a start state and an accept state allow automata to do something the semiautomata cannot: they can recognize a formal language. The language L accepted by a deterministic finite automaton $\langle Q, \Sigma, \delta, q_0, F \rangle$ is:

$$L = \{w \in \Sigma^\star \mid \widehat{\delta}(q_0, w) \in F\}$$

That is, the language accepted by an automaton is the set of all words *w*, over the alphabet Σ, that, when given as input to the automaton, will result in its ending in some state from F. Languages that are accepted by automata are called recognizable languages.

When the set of states *Q* is finite, then the automaton is known as a finite state automaton, and the set of all recognizable languages are the regular languages. In fact, there is a strong equivalence: for every regular language, there is a finite state automaton, and *vice versa*.

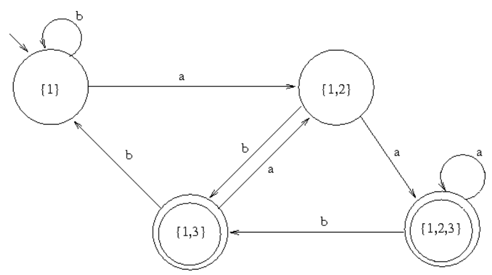
As noted above, the set *Q* need not be finite or countable; it may be taken to be a general topological space, in which case one obtains topological automata. Another possible generalization is the metric automata or *geometric automata*. In this case, the acceptance of a language is altered: instead of a set inclusion of the final state in $\widehat{\delta}(q_0, w) \in F$, the acceptance criteria are replaced by a probability, given in terms of the metric distance between the final state $\widehat{\delta}(q_0, w)$and the set F. Certain types of probabilistic automata are metric automata, with the metric being a measure on a probability space.

Classes of finite automata

The following are three kinds of finite automata

Deterministic finite automata (DFA)

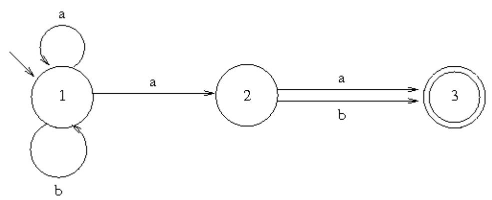
Each state of an automaton of this kind has a transition for every symbol in the alphabet.



DFA

Nondeterministic finite automata (NFA)

States of an automaton of this kind may or may not have a transition for each symbol in the alphabet, or can even have multiple transitions for a symbol. The automaton accepts a word if there exists at least one path from q_0 to a state in F *labeled* with the input word. If a transition is *undefined*, so that the automaton does not know how to keep on reading the input, the word is rejected.



NFA, equivalent to the DFA from the previous example

Nondeterministic finite automata, with ε transitions (FND-ε or ε-NFA)

Besides of being able to jump to more (or none) states with any symbol, these can jump on no symbol at all. That is, if a state has transitions labeled with ϵ, then the NFA can *be* in any of the states reached by the ϵ-transitions, directly or through other states with ϵ-transitions. The set of states that can be reached by this method from a state q, is called the ϵ-closure of q.

It can be shown, though, that all these automata **can accept the same languages**. You can always construct some DFA M' that accepts the same language as a given NFA M.

Extensions of finite automata

The family of languages accepted by the above-described automata is called the family of regular languages. More powerful automata can accept more complicated languages. Such automata include:

Pushdown automata (PDA)

Such machines are identical to DFAs (or NFAs), except that they additionally carry memory in the form of a stack. The transition function δ will now also depend on the symbol(s) on top of the stack, and will specify how the stack is to be changed at each transition. Non-determinstic PDAs accept the context-free languages.

Linear Bounded Automata (LBA)

An LBA is a limited Turing machine; instead of an infinite tape, the tape has an amount of space proportional to the size of the input string. LBAs accept the context-sensitive languages.

Turing machines

These are the most powerful computational machines. They possess an infinite memory in the form of a tape, and a head which can read and change the tape, and move in either direction along the tape. Turing machines are equivalent to algorithms, and are the theoretical basis for modern computers. Turing machines decide/accept recursive languages and recognize the recursively enumerable languages.

Timed automata

Automata, where timing plays a crucial role in the question of correctness. Timed automata work with *timed* sequences of events, opposite to normal automata.....

External links

- Visual Automata Simulator [2], A tool for simulating, visualizing and transforming finite state automata and Turing Machines, by Jean Bovet
- JFLAP [3]
- dk.brics.automaton [4]
- libfa [5]
- Proyecto SEPa (in Spanish) [6]
- Exorciser (in German) [7]

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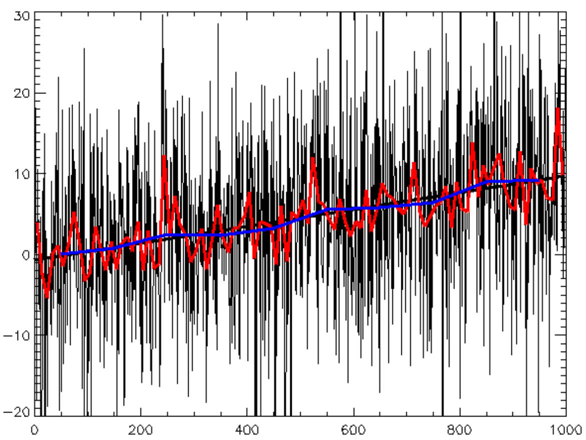
Time series analysis

In statistics, signal processing, and many other fields, a **time series** is a sequence of data points, measured typically at successive times, spaced at (often uniform) time intervals. **Time series *analysis*** comprises methods that attempt to understand such time series, often either to understand the underlying context of the data points (Where did they come from? What generated them?), or to make forecasts (predictions). **Time series *forecasting*** is the use of a model to forecast future events based on known past events: to forecast future data points before they are measured. A standard example in econometrics is the opening price of a share of stock based on its past performance.



Time series: random data plus trend, with best-fit line and different smoothings

The term time series analysis is used to distinguish a problem, firstly from more ordinary data analysis problems (where there is no natural ordering of the context of individual observations), and secondly from spatial data analysis where there is a context that observations (often) relate to geographical locations. There are additional possibilities in the form of space-time models (often called spatial-temporal analysis). A time series model will generally reflect the fact that observations close together in time will be more closely related than observations further apart. In addition, time series models will often make use of the natural one-way ordering of time so that values in a series for a given time will be expressed as deriving in some way from past values, rather than from future values (see time reversibility.)

Methods for time series analyses are often divided into two classes: frequency-domain methods and time-domain methods. The former centre around spectral analysis and recently wavelet analysis, and can be regarded as model-free analyses well-suited to exploratory investigations. Time-domain methods have a model-free subset consisting of the examination of auto-correlation and cross-correlation analysis, but it is here that partly and fully-specified time series models make their appearance.

Analysis

There are several types of data analysis available for time series which are appropriate for different purposes.

General exploration

- Graphical examination of data series
- Autocorrelation analysis to examine serial dependence
- Spectral analysis to examine cyclic behaviour which need not be related to seasonality

Description

- Separation into components representing trend, seasonality, slow and fast variation, cyclical irregular: see Decomposition of time series
- Simple properties of marginal distributions

Prediction and forecasting

- Fully-formed statistical models for stochastic simulation purposes, so as to generate alternative versions of the time series, representing what might happen over non-specific time-periods in the future (prediction).
- Simple or fully-formed statistical models to describe the likely outcome of the time series in the immediate future, given knowledge of the most recent outcomes (forecasting).

Models

Models for time series data can have many forms and represent different stochastic processes. When modeling variations in the level of a process, three broad classes of practical importance are the *autoregressive* (AR) models, the *integrated* (I) models, and the *moving average* (MA) models. These three classes depend linearly[1] on previous data points. Combinations of these ideas produce autoregressive moving average (ARMA) and autoregressive integrated moving average (ARIMA) models. The autoregressive fractionally integrated moving average (ARFIMA) model generalizes the former three. Extensions of these classes to deal with vector-valued data are available under the heading of multivariate time-series models and sometimes the preceding acronyms are extended by including an initial "V" for "vector". An additional set of extensions of these models is available for use where the observed time-series is driven by some "forcing" time-series (which may not have a causal effect on the observed series): the distinction from the multivariate case is that the forcing series may be deterministic or under the experimenter's control. For these models, the acronyms are extended with a final "X" for "exogenous".

Non-linear dependence of the level of a series on previous data points is of interest, partly because of the possibility of producing a chaotic time series. However, more importantly, empirical investigations can indicate the advantage of using predictions derived from non-linear models, over those from linear models.

Among other types of non-linear time series models, there are models to represent the changes of variance along time (heteroskedasticity). These models are called autoregressive conditional heteroskedasticity (ARCH) and the collection comprises a wide variety of representation (GARCH, TARCH, EGARCH, FIGARCH, CGARCH, etc). Here changes in variability are related to, or predicted by, recent past values of the observed series. This is in contrast to other possible representations of locally-varying variability, where the variability might be modelled as being driven by a separate time-varying process, as in a doubly stochastic model.

In recent work on model-free analyses, wavelet transform based methods (for example locally stationary wavelets and wavelet decomposed neural networks) have gained favor. Multiscale (often referred to as multiresolution) techniques decompose a given time series, attempting to illustrate time dependence at multiple scales.

Notation

A number of different notations are in use for time-series analysis:

$$X = \{X_1, X_2, ...\}$$

is a common notation which specifies a time series *X* which is indexed by the natural numbers. Another common notation is:

$$Y = \{Y_t: t \in T\}.$$

Conditions

There are two sets of conditions under which much of the theory is built:

- Stationary process
- Ergodicity

However, ideas of stationarity must be expanded to consider two important ideas: strict stationarity and second-order stationarity. Both models and applications can be developed under each of these conditions, although the models in the latter case might be considered as only partly specified.

In addition, time-series analysis can be applied where the series are seasonally stationary and non-stationary.

Models

The general representation of an autoregressive model, well-known as AR(*p*), is

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \cdots + \alpha_p Y_{t-p} + \varepsilon_t$$

where the term ε_t is the source of randomness and is called white noise. It is assumed to have the following characteristics:

1. $E[\varepsilon_t] = 0$
2. $E[\varepsilon_t^2] = \sigma^2$
3. $E[\varepsilon_t \varepsilon_s] = 0 \quad \forall t \neq s$

With these assumptions, the process is specified up to second-order moments and, subject to conditions on the coefficients, may be second-order stationary.

If the noise also has a normal distribution, it is called normal white noise (denoted here by Normal-WN):

$$\{\varepsilon_t\}_{(t \in T)} : \text{Normal-WN}.$$

In this case the AR process may be strictly stationary, again subject to conditions on the coefficients.

Related tools

Tools for investigating time-series data include:

- Consideration of the autocorrelation function and the spectral density function (also cross-correlation functions and cross-spectral density functions)
- Performing a Fourier transform to investigate the series in the frequency domain.
- Use of a filter to remove unwanted noise.
- Principal components analysis (or empirical orthogonal function analysis)
- Singular spectrum analysis
- Artificial neural networks
- time-frequency analysis techniques:
  - Continuous wavelet transform
  - Short-time Fourier transform

  - Chirplet transform
  - Fractional Fourier transform
- Chaotic analysis
  - Correlation dimension
  - Recurrence plots
  - Recurrence quantification analysis
  - Lyapunov exponents

See also

- Analysis of rhythmic variance
- Anomaly time series
- Autocorrelation
- Partial autocorrelation
- Linear prediction
- Longitudinal study
- Model (macroeconomics)
- Moving average
- Nonlinear autoregressive exogenous model
- Prediction interval
- Seasonal adjustment
- System identification
- Time series database
- Trend estimation

References

- Box, George; Jenkins, Gwilym (1976), *Time series analysis: forecasting and control, rev. ed.*, Oakland, California: Holden-Day
- Gershenfeld, Neil (2000), *The nature of mathematical modeling*, Cambridge: Cambridge Univ. Press, ISBN 978-0521570954, OCLC 174825352 [2]

External links

- A First Course on Time Series Analysis [3] - an open source book on time series analysis with SAS
- Introduction to Time series Analysis (Engineering Statistics Handbook) [4] - A practical guide to Time series analysis
- List of Free Software for Time Series Analysis [5]
- Online Tutorial 'Recurrence Plot' (Flash animation); lots of examples [6]

References

[1] linear time series: It is interesting to note the "breakdown of linear systems theory". Gershenfeld 1999, p.205-08
[2] http://worldcat.org/oclc/174825352
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[4] http://www.itl.nist.gov/div898/handbook/pmc/section4/pmc4.htm
[5] http://ces.stat.ucla.edu/software/time-series-analysis
[6] http://www.as-internetdienst.de/r67tze4/einbettung.html

Lag operator

In time series analysis, the **lag operator** or **backshift operator** operates on an element of a time series to produce the previous element. For example, given some time series

$$X = \{X_1, X_2, \ldots\}$$

then

$$LX_t = X_{t-1} \text{for all } t > 1$$

where L is the lag operator. Sometimes the symbol B for backshift is used instead. Note that the lag operator can be raised to arbitrary integer powers so that

$$L^{-1}X_t = X_{t+1}$$

and

$$L^k X_t = X_{t-k}.$$

Lag polynomials

Also polynomials of the lag operator can be used, and this is a common notation for ARMA models. For example,

$$\varepsilon_t = X_t - \sum_{i=1}^{p} \varphi_i X_{t-i} = \left(1 - \sum_{i=1}^{p} \varphi_i L^i\right) X_t$$

specifies an AR(p) model.

A polynomial of lag operators is called a **lag polynomial** so that, for example, the ARMA model can be concisely specified as

$$\varphi X_t + \theta \varepsilon_t$$

where φ and θ respectively represent the lag polynomials,

$$\varphi = 1 - \sum_{i=1}^{p} \varphi_i L^i$$

and

$$\theta = 1 + \sum_{i=1}^{q} \theta_i L^i.$$

An **annihilator operator**, denoted $[\ ]_+$, removes the entries of the polynomial with negative power (future values).

Difference operator

In time series analysis, the first difference operator Δ is a special case of lag polynomial.

$$\begin{aligned} \Delta X_t &= X_t - X_{t-1} \\ \Delta X_t &= (1-L)X_t \end{aligned}$$

Similarly, the second difference operator

$$\begin{aligned} \Delta(\Delta X_t) &= \Delta X_t - \Delta X_{t-1} \\ \Delta^2 X_t &= (1-L)\Delta X_t \\ \Delta^2 X_t &= (1-L)(1-L)X_t \\ \Delta^2 X_t &= (1-L)^2 X_t \end{aligned}$$

The above approach generalises to the *i* 'th difference operator $\Delta^i X_t = (1-L)^i X_t$

Conditional Expectation

It is common in stochastic processes to care about the expected value of a variable given a previous information set. Let Ω_t be all information that is common knowledge at time *t* (this is often subscripted below the expectation operator), then the expected value of *X* that is some *j* time-steps in the future can be written equivalently as:

$$E[X_{t+j}|\Omega_t] = E_t[X_{t+j}] \, .$$

With these time-dependent conditional expectations, there is the need to distinguish between the Backshift operator (*B*) that only adjusts the date of the forecasted variable and the Lag operator (*L*) that adjusts equally the date of the forecasted variable and the information set:

$$\begin{aligned} L^n E_t[X_{t+j}] &= E_{t-n}[X_{t+j-n}] \, , \\ B^n E_t[X_{t+j}] &= E_t[X_{t+j-n}] \, . \end{aligned}$$

See also

- Autoregressive model
- Autoregressive moving average model
- → Shift operator
- Z-transform

Shift operator

In mathematics, and in particular functional analysis, the **shift operators** are examples of linear operators, important for their simplicity and natural occurrence. They are used in diverse areas, such as Hardy spaces, the theory of abelian varieties, and the theory of → symbolic dynamics, for which the baker's map is an explicit representation. (There is another usage of *shift operator* as a translation operator: see for example Sheffer sequence.) In → time series analysis, this operator is called the → **lag operator**.

A typical **one-sided shift** operator takes an infinite sequence of numbers

$(a_1, a_2, ...)$

to

$(0, a_1, a_2, ...)$.

This operation respects typical convergence conditions, such as absolute convergence of the corresponding infinite series; it therefore gives rise to continuous operators on the standard sequence spaces used in functional analysis, usually with norm 1.

Another way to look at it would be in terms of polynomials: the sequences that eventually end in a string

$(..., 0, 0, 0, ...)$

or, in other words, having only a finite number of non-zero entries, are in a 1-1 correspondence with polynomials in an indeterminate T having a_i as coefficient of T^i. The advantage of this representation is then that the *shift operator* becomes multiplication by T: this reveals quickly several aspects of its structure. Spaces of polynomials carry numerous topological structures; shift operators can be constructed by extension on corresponding complete spaces.

The **bilateral shift** operators are the related operators in which the sequences are bi-infinite (functions on the integers, rather than just the natural numbers). One can say that the analogue in this case of the polynomial representation is that by Laurent polynomials. The theory of analytic functions is related to that of polynomials, by allowing infinite power series; on the other hand meromorphic functions have Laurent series that terminate in the direction of negative exponents. In the same way, the one-sided and bilateral shifts have rather different properties. This connection with function theory is made more precise in the context of Hardy spaces.

Action on Hilbert spaces

The unilateral and bilateral shifts have a natural action on → Hilbert spaces, giving bounded operators S and T on the ℓ^p sequence spaces $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ respectively. The unilateral shift S is a proper isometry with range equal to all vectors which vanish in the first coordinate. The bilateral shift U, on the other hand, is a unitary operator. The operator S is a compression of U, in the sense that

$$Ux' = Sx \text{ for each } x \in \ell^2(\mathbb{N}),$$

where x' is the vector in $\ell^2(\mathbb{Z})$ with $x'_i = x_i$ for $i \geq 0$ and $x'_i = 0$ for $i < 0$. This observation is at the heart of the construction of many unitary dilations of isometries.

The spectrum of S is the unit disk while the spectrum of U is the unit circle in the complex plane.

The Wold decomposition says that every isometry on a Hilbert space is of the form

$$S^\alpha \oplus U$$

where S^α is S to the power of some cardinal number α and U is a unitary operator. In turn, the C*-algebra generated by an arbitrary proper isometry is isomorphic to the C*-algebra generated by S.

The shift S is one example of a Fredholm operator; it has Fredholm index −1.

See also

- Dilation
- Arithmetic shift
- Logical shift

Shift space

In → symbolic dynamics and related branches of mathematics, a **shift space** or **subshift** is a set of infinite words representing the evolution of a discrete system. In fact, shift spaces and → *symbolic dynamical systems* are often considered synonyms.

Notation

Let *A* be a finite set of states. An *infinite* (respectively *bi-infinite*) *word* over *A* is a sequence $\mathbf{x} = (x_n)_{n\in M}$, where $M = \mathbb{N}$ (resp. $M = \mathbb{Z}$) and x_n is in *A* for any integer *n*. The → shift operator acts on an infinite or bi-infinite word by shifting all symbols to the left, i.e.,

$(\sigma(\mathbf{x}))(n) = x_{n+1}$ for all *n*.

In the following we choose $M = \mathbb{N}$ and thus speak of infinite words, but all definitions are naturally generalizable to the bi-infinite case.

Definition

A set of infinite words over *A* is a *shift space* if it is closed with respect to the natural product topology of $A^{\mathbb{N}}$ and invariant under the shift operator. Thus a set $S \subseteq A^{\mathbb{N}}$ is a subshift if and only if

1. for any (pointwise) convergent sequence $(\mathbf{x}_k)_{k\geq 0}$ of elements of *S*, the limit $\lim_{k\to\infty} \mathbf{x}_k$ also belongs to *S*; and
2. $\sigma(X) = X$.

A shift space *S* is sometimes denoted as (S, σ) in order to emphasize the role of the shift operator.
Some authors[1] use the term *subshift* for a set of infinite words which is just invariant under the shift, and reserve the term *shift space* for those which are also closed.

Characterization and sofic subshifts

A subset *S* of $A^{\mathbb{N}}$ is a shift space if and only if there exists a set *X* of finite words such that *S* coincides with the set of all infinite words over *A* having no factor in *X*.
When *X* is a regular language, the corresponding subshift is called **sofic**. In particular, if *X* is finite then *S* is called a subshift of finite type.

Examples

The first trivial example of shift space (of finite type) is the *full shift* $A^{\mathbb{N}}$.

Let $A = \{a, b\}$. The set of all infinite words over *A* containing at most one *b* is a sofic subshift, not of finite type.

Further reading

- Lind, Douglas; Marcus, Brian (1995). *An Introduction to Symbolic Dynamics and Coding*. Cambridge UK: Cambridge University Press. ISBN 0521559006.
- Lothaire, M. (2002). "Finite and Infinite Words [2]". *Algebraic Combinatorics on Words* [3]. Cambridge UK: Cambridge University Press. ISBN 0521812208. Retrieved 2008-01-29.
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Markov partition

Markov partition is a fundamental concept in the mathematical theory of dynamical systems which allows one to represent a discrete dynamical system as a shift of finite type on an auxiliary space of sequences of abstract symbols. Such a partition shows that, at a coarse level, the deterministic dynamic system resembles a discrete-time Markov process and allows to apply methods of → symbolic dynamics to the study of long-term dynamical characteristics of the system, such as its topological entropy.

Motivation

Let (M,φ) be a discrete dynamical system. A basic method of studying its dynamics is to find a **symbolic representation**: a faithful encoding of the points of M by sequences of symbols such that the map φ becomes the shift map.

Suppose that M has been divided into a number of pieces $E_1,E_2,\ldots,E_r$, which are thought to be as small and localized, with virtually no overlaps. The behavior of a point x under the iterates of φ can be tracked by recording, for each n, the part E_i which contains $\varphi^n(x)$. This results in an infinite sequence on the alphabet $\{1,2,\ldots r\}$ which encodes the point. In general, this encoding may be imprecise (the same sequence may represent many different points) and the set of sequences which arise in this way may be difficult to describe. Under certain conditions, which are made explicit in the rigorous definition of a Markov partition, the assignment of the sequence to a point of M becomes an almost one-to-one map whose image is a symbolic dynamical system of a special kind called a shift of finite type. In this case, the symbolic representation is a powerful tool for investigating the properties of the dynamical system (M,φ).

Examples

Markov partitions have been constructed in several situations.

- Anosov diffeomorphisms of the torus.
- Dynamical billiards.

References

- Douglas Lind and Brian Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, 1995 ISBN 0-521-55124-2

Sharkovskii's theorem

In mathematics, **Sharkovskii's theorem** is a result about discrete dynamical systems. It is named for Oleksandr Mikolaiovich Sharkovsky. One of the implications of the theorem is that if a continuous discrete dynamical system on the real line has a periodic point of period 3, then it must have periodic points of every other period.

The theorem

Suppose

$$f : \mathbf{R} \to \mathbf{R}$$

is a continuous function. We say that the number x is a *periodic point of period m* if $f^m(x) = x$ (where f^m denotes the composition of m copies of f) and having *least period m* if furthermore $f^k(x) \neq x$ for all $0 < k < m$. We are interested in the possible periods of periodic points of f. Consider the following ordering of the positive integers:

$$\begin{array}{l}
3, 5, 7, 9, 11, \ldots, (2n+1)\cdot 2^0, \ldots \\
2\cdot 3, 2\cdot 5, 2\cdot 7, 2\cdot 9, 2\cdot 11, \ldots, (2n+1)\cdot 2^1, \ldots \\
2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, 2^2\cdot 9, 2^2\cdot 11, \ldots, (2n+1)\cdot 2^2, \ldots \\
2^3\cdot 3, 2^3\cdot 5, 2^3\cdot 7, 2^3\cdot 9, 2^3\cdot 11, \ldots, (2n+1)\cdot 2^3, \ldots \\
\qquad\qquad\vdots \\
\ldots, 2^n, \ldots, 2^5, 2^4, 2^3, 2^2, 2, 1.
\end{array}$$

We start, that is, with the odd numbers in increasing order, then 2 times the odds, 4 times the odds, 8 times the odds, etc., and at the end we put the powers of two in decreasing order. Sharkovskii's theorem states that if f has a periodic point of least period m and $m \leq n$ in the above ordering, then f has also a periodic point of least period n.

As a consequence, we see that if f has only finitely many periodic points, then they must all have periods which are powers of two. Furthermore, if there is a periodic point of period three, then there are periodic points of all other periods.

Sharkovskii's theorem does not state that there are *stable* cycles of those periods, just that there are cycles of those periods. For systems such as the logistic map, the bifurcation diagram shows a range of parameter values for which apparently the only cycle has period 3. In fact, there must be cycles of all periods there, but they are not stable and therefore not visible on the computer generated picture.

Interestingly, the above "Sharkovskii ordering" of the positive integers also occurs in a slightly different context in connection with the logistic map: the *stable* cycles appear in this order in the bifurcation diagram, starting with 1 and ending with 3, as the parameter is increased. (Here we ignore a stable cycle if a stable cycle of the same order has occurred earlier.)

The assumption of continuity is important, as the discontinuous function $f : x \to (1-x)^{-1}$, for which every value has period 3, would otherwise be a counterexample.

Generalizations

Sharkovskii's theorem does not immediately apply to dynamical systems on other topological spaces. It is easy to find a circle map with periodic points of period 3 only: take a rotation by 120 degrees, for example. But some generalizations are possible, typically involving the mapping class group of the space minus a periodic orbit.

References

- Weisstein, Eric W., "Sharkovskys Theorem [1]" from MathWorld.
- Sharkovskii's theorem [2] on PlanetMath

References

[1] http://mathworld.wolfram.com/SharkovskysTheorem.html
[2] http://planetmath.org/?op=getobj&from=objects&id=3751

Ergodic system

Ergodic theory is a branch of mathematics that studies → dynamical systems with an invariant measure and related problems. Its initial development was motivated by problems of statistical physics.

A central aspect of ergodic theory is the behavior of a dynamical system when it is allowed to run long. This is expressed through *ergodic theorems* which assert that, under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average. Two most important examples are the ergodic theorems of Birkhoff and von Neumann. For the special class of *ergodic systems*, the time average is the same for almost all initial points: statistically speaking, the system that evolves for a long time "forgets" its initial state. Stronger properties, such as mixing and equidistribution have also been extensively studied. The problem of metric classification of systems is another important part of the abstract ergodic theory. An outstanding role in ergodic theory and its applications to stochastic processes is played by the various notions of entropy for dynamical systems.

Applications of ergodic theory to other parts of mathematics usually involve establishing ergodicity properties for systems of special kind. In geometry, methods of ergodic theory have been used to study the geodesic flow on Riemannian manifolds, starting with the results of Eberhard Hopf for Riemann surfaces of negative curvature. Markov chains form a common context for applications in probability theory. Ergodic theory has fruitful connections with harmonic analysis, Lie theory (representation theory, lattices in algebraic groups), and number theory (the theory of diophantine approximations, L-functions).

Ergodic transformations

Let T: $X \to X$ be a measure-preserving transformation on a measure space (X, Σ, μ), usually assumed to have finite measure. An element A of Σ is ***T*-invariant mod 0** if $T^{-1}(A)$ differs from A by a set of measure zero:

$$\mu(T^{-1}(A) \triangle A) = 0,$$

where $\triangle$ denotes the symmetric difference. If this is true then A is T^n-invariant mod 0 for all n.
A measure-preserving transformation T as above is **ergodic** if for every T-invariant element mod 0 measurable set A, either A or its complement $X \setminus A$ has measure zero. In older literature, ergodic transformations were called **metrically transitive**.

These definitions have natural analogues for the case of measurable flows and, more generally, measure-preserving semigroup actions. Let $\{T^t\}$ be a measurable flow on (X, Σ, μ). An element A of Σ is invariant mod 0 under $\{T^t\}$ if

$$\mu(T^t(A) \triangle A) = 0$$

for each $t \in \mathbf{R}$. Measurable sets invariant mod 0 under a flow or a semigroup action form the **invariant subalgebra** of Σ, and the corresponding → measure-preserving dynamical system is ergodic if the invariant subalgebra is the trivial σ-algebra consisting of the sets of measure 0 and their complements in X. If the measure is *normalized*, $\mu(X)=1$, so that (X, Σ, μ) is a probability space, then all invariant mod 0 sets must have measure 0 or 1.

Conceptually, ergodicity of a dynamical system is a certain irreducibility property, akin to the notions of irreducible representation in algebra and prime number in arithmetic. A general measure-preserving transformation or flow on a Lebesgue space admits a canonical decomposition into its **ergodic components**, each of which is ergodic.

Examples

- An irrational rotation of the circle **R/Z**, T: $x \to x+\theta$, where θ is irrational, is ergodic. This transformation has even stronger properties of unique ergodicity, minimality, and equidistribution. By contrast, if $\theta = p/q$ is rational (in lowest terms) then T is periodic, with period q, and thus cannot be ergodic: for any interval I of length a, $0 < a < 1/q$, its orbit under T is a T-invariant mod 0 set that is a union of q intervals of length a, hence it has measure qa strictly between 0 and 1.
- Let G be a compact abelian group, μ the normalized Haar measure, and T a group automorphism of G. Let G^* be the Pontryagin dual group, consisting of the continuous characters of G, and T^* be the corresponding adjoint automorphism of G^*. The automorphism T is ergodic if and only if the equality $(T^*)^n(\chi)=\chi$ is possible only when $n = 0$ or χ is the trivial character of G. In particular, if G is the n-dimensional torus and the automorphism T is represented by an integral matrix A then T is ergodic if and only if no eigenvalue of A is a root of unity.
- A Bernoulli shift is ergodic. More generally, ergodicity of the shift transformation associated with a sequence of i.i.d. random variables and some more general stationary processes follows from Kolmogorov's zero-one law.
- Ergodicity of a continuous dynamical system means that its trajectories "spread around" the phase space. A system with a compact phase space which has a non-constant first integral cannot be ergodic. This applies, in particular, to Hamiltonian systems with a first integral I functionally independent from the Hamilton function H and a compact level set $X = \{(p,q): H(p,q)=E\}$ of constant energy. Liouville's theorem implies the existence of a finite invariant measure on X, but the dynamics of the system is constrained to the level sets of I on X, hence the system possesses invariant sets of positive but less than full measure. A property of continuous dynamical systems that is the opposite of ergodicity is complete integrability.

Ergodic theorems

Let $T: X \to X$ be a measure-preserving transformation on a measure space (X, Σ, μ). One may then consider the "time average" of a μ-integrable function *f*, i.e. $f \in L^1(\mu)$. The "time average" is defined as the average (if it exists) over iterations of *T* starting from some initial point *x*.

$$\hat{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x\right).$$

If $\mu(X)$ is finite and nonzero, we can consider the "space average" or "phase average" of *f*, defined as

$$\bar{f} = \frac{1}{\mu(X)} \int f \, d\mu.$$ (For a probability space, $\mu(X) = 1$)

In general the time average and space average may be different. But if the transformation is ergodic, and the measure is invariant, then the time average is equal to the space average almost everywhere. This is the celebrated ergodic theorem, in an abstract form due to George David Birkhoff. (Actually, Birkhoff's paper considers not the abstract general case but only the case of dynamical systems arising from differential equations on a smooth manifold.) The equidistribution theorem is a special case of the ergodic theorem, dealing specifically with the distribution of probabilities on the unit interval.

More precisely, the **pointwise** or **strong ergodic theorem** states that the limit in the definition of the time average of *f* exists for almost every *x* and that the (almost everywhere defined) limit function $\hat{f}$ is integrable:

$$\hat{f} \in L^1(\mu).$$

Furthermore, $\hat{f}$ is *T*-invariant, that is to say

$$\hat{f} \circ T = \hat{f}$$

holds almost everywhere, and if $\mu(X)$ is finite, then the normalization is the same:

$$\int \hat{f} \, d\mu = \int f \, d\mu.$$

In particular, if *T* is ergodic, then $\hat{f}$ must be a constant (almost everywhere), and so one has that

$$\bar{f} = \hat{f}$$

almost everywhere. Joining the first to the last claim and assuming that $\mu(X)$ is finite and nonzero, one has that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x\right) = \frac{1}{\mu(X)} \int f \, d\mu$$

for almost all *x*, i.e., for all *x* except for a set of measure zero.

For an ergodic transformation, the time average equals the space average almost surely.

As an example, assume that the measure space (X, Σ, μ) models the particles of a gas as above, and let *f*(*x*) denotes the velocity of the particle at position *x*. Then the pointwise ergodic theorems says that the average velocity of all particles at some given time is equal to the average velocity of one particle over time.

Probabilistic formulation: Birkhoff-Khinchin theorem

Birkhoff-Khinchin theorem. Let f be measurable, $E(|f|) < +\infty$, and T be a measure-preserving operator. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(T^k x\right) = E(f|\mathcal{C}),$$

where $E(f|\mathcal{C})$ is the conditional expectation given the σ -algebra $\mathcal{C}$ of invariant sets of T.

Corollary (**Pointwise ergodic theorem**) In particular, if T is also ergodic, then $\mathcal{C}$ is the trivial σ -algebra, and thus

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(T^k x\right) = E(f) \text{ a.s.}$$

Mean ergodic theorem

Another form of the ergodic theorem, **von Neumann's mean ergodic theorem**, holds in Hilbert spaces.[1]

Let U be a unitary operator on a → Hilbert space H. Let P be the orthogonal projection onto $\{\psi \in H | U\psi = \psi\} = \mathrm{Ker}(\mathrm{id} - U)$.

Then, for any $x \in H$, we have:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n x = Px,$$

where the limit is with respect to the norm on *H*. In other words, the sequence of averages

$$\frac{1}{N}\sum_{n=0}^{N-1} U^n$$

converges to *P* in the strong operator topology.

This theorem specializes to the case in which the Hilbert space *H* consists of L^2 functions on a measure space and *U* is an operator of the form

$$Uf(x) = f(Tx)$$

where *T* is a measure-preserving automorphism of *X*, thought of in applications as representing a time-step of a discrete dynamical system.[2] The ergodic theorem then asserts that the average behavior of a function *f* over sufficiently large time-scales is approximated by the orthogonal component of *f* which is time-invariant.

In another form of the mean ergodic theorem, let U_t be a strongly continuous one-parameter group of unitary operators on *H*. Then the operator

$$\frac{1}{T}\int_0^T U_t\,dt$$

converges in the strong operator topology as $T \to \infty$. In fact, this result also extends to the case of strongly continuous one-parameter semigroup of contractive operators on a reflexive space.

Remark: Some intuition for the mean ergodic theorem can be developed by considering the case where complex numbers of unit length are regarded as unitary transformations on the complex plane (by left multiplication). If we pick a single complex number of unit length (which we think of as U), it is intuitive that its powers will fill up the circle. Since the circle is symmetric around 0, it makes sense that the averages of the powers of U will converge to 0. Also, 0 is the only fixed point of U , and so the the projection onto the space of fixed points must be the zero operator (which agrees with the limit just described).

Sojourn time

Let (X, Σ, μ) be a measure space such that $\mu(X)$ is finite and nonzero. The time spent in a measurable set *A* is called the **sojourn time**. An immediate consequence of the ergodic theorem is that, in an ergodic system, the relative measure of *A* is equal to the mean sojourn time:

$$\frac{\mu(A)}{\mu(X)} = \frac{1}{\mu(X)} \int \chi_A \, d\mu = \lim_{n\to\infty} \; \frac{1}{n} \sum_{k=0}^{n-1} \chi_A \left(T^k x\right)$$

where χ_A is the indicator function of *A*, for all *x* except for a set of measure zero.

Let the **occurrence times** of a measurable set *A* be defined as the set k_1, k_2, k_3, ..., of times *k* such that $T^k(x)$ is in *A*, sorted in increasing order. The differences between consecutive occurrence times $R_i = k_i - k_{i-1}$ are called the **recurrence times** of *A*. Another consequence of the ergodic theorem is that the average recurrence time of *A* is inversely proportional to the measure of *A*, assuming that the initial point *x* is in *A*, so that $k_0 = 0$.

$$\frac{R_1 + \cdots + R_n}{n} \rightarrow \frac{\mu(X)}{\mu(A)} \quad \text{(almost surely)}$$

(See almost surely.) That is, the smaller *A* is, the longer it takes to return to it.

Ergodic flows on manifolds

The ergodicity of the geodesic flow on compact Riemann surfaces of variable negative curvature and on compact manifolds of constant negative curvature of any dimension was proved by Eberhard Hopf in 1939, although special cases had been studied earlier: see for example, Hadamard's billiards (1898) and Artin billiard (1924). The relation between geodesic flows on Riemann surfaces and one-parameter subgroups on SL(2,**R**) was described in 1952 by S. V. Fomin and I. M. Gelfand. The article on Anosov flows provides an example of ergodic flows on SL(2,**R**) and on Riemann surfaces of negative curvature. Much of the development described there generalizes to hyperbolic manifolds, since they can be viewed as quotients of the hyperbolic space by the action of a lattice in the semisimple Lie group SO(n,1). Ergodicity of the geodesic flow on Riemannian symmetric spaces was demonstrated by F. I. Mautner in 1957. In 1967 D. V. Anosov and Ya. G. Sinai proved ergodicity of the geodesic flow on compact manifolds of variable negative sectional curvature. A simple criterion for the ergodicity of a homogeneous flow on a homogeneous space of a semisimple Lie group was given by C. C. Moore in 1966. Many of the theorems and results from this area of study are typical of rigidity theory.

In the 1930s → G. A. Hedlund proved that the horocycle flow on a compact hyperbolic surface is minimal and ergodic. Unique ergodicity of the flow was established by Hillel Furstenberg in 1972. Ratner's theorems provide a major generalization of ergodicity for unipotent flows on the homogeneous spaces of the form $\Gamma\backslash G$, where *G* is a Lie group and *Γ* is a lattice in *G*.

See also

- Chaos theory
- → Dynamical systems theory
- Ergodic hypothesis
- Ergodic process
- Functional analysis
- Maximal ergodic theorem
- Poincaré recurrence theorem
- Statistical mechanics
- Markov chain

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Ergodic theory

Ergodic theory is a branch of mathematics that studies → dynamical systems with an invariant measure and related problems. Its initial development was motivated by problems of statistical physics.

A central aspect of ergodic theory is the behavior of a dynamical system when it is allowed to run long. This is expressed through *ergodic theorems* which assert that, under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average. Two most important examples are the ergodic theorems of Birkhoff and von Neumann. For the special class of *ergodic systems*, the time average is the same for almost all initial points: statistically speaking, the system that evolves for a long time "forgets" its initial state. Stronger properties, such as mixing and equidistribution have also been extensively studied. The problem of metric classification of systems is another important part of the abstract ergodic theory. An outstanding role in ergodic theory and its applications to stochastic processes is played by the various notions of entropy for dynamical systems.

Applications of ergodic theory to other parts of mathematics usually involve establishing ergodicity properties for systems of special kind. In geometry, methods of ergodic theory have been used to study the geodesic flow on Riemannian manifolds, starting with the results of Eberhard Hopf for Riemann surfaces of negative curvature. Markov chains form a common context for applications in probability theory. Ergodic theory has fruitful connections with harmonic analysis, Lie theory (representation theory, lattices in algebraic groups), and number theory (the theory of diophantine approximations, L-functions).

Ergodic transformations

Let T: $X \to X$ be a measure-preserving transformation on a measure space (X, Σ, μ), usually assumed to have finite measure. An element A of Σ is ***T*-invariant mod 0** if $T^{-1}(A)$ differs from A by a set of measure zero:

$$\mu(T^{-1}(A) \triangle A) = 0,$$

where $\triangle$ denotes the symmetric difference. If this is true then A is T^n-invariant mod 0 for all n.
A measure-preserving transformation T as above is **ergodic** if for every T-invariant element mod 0 measurable set A, either A or its complement $X\backslash A$ has measure zero. In older literature, ergodic transformations were called **metrically transitive**.

These definitions have natural analogues for the case of measurable flows and, more generally, measure-preserving semigroup actions. Let $\{T^t\}$ be a measurable flow on (X, Σ, μ). An element A of Σ is invariant mod 0 under $\{T^t\}$ if

$$\mu(T^t(A) \triangle A) = 0$$

for each $t \in \mathbf{R}$. Measurable sets invariant mod 0 under a flow or a semigroup action form the **invariant subalgebra** of Σ, and the corresponding → measure-preserving dynamical system is ergodic if the invariant subalgebra is the trivial σ-algebra consisting of the sets of measure 0 and their complements in X. If the measure is *normalized*, $\mu(X)=1$, so that (X, Σ, μ) is a probability space, then all invariant mod 0 sets must have measure 0 or 1.

Conceptually, ergodicity of a dynamical system is a certain irreducibility property, akin to the notions of irreducible representation in algebra and prime number in arithmetic. A general measure-preserving transformation or flow on a Lebesgue space admits a canonical decomposition into its **ergodic components**, each of which is ergodic.

Examples

- An irrational rotation of the circle $\mathbf{R}/\mathbf{Z}$, T: $x \to x+\theta$, where θ is irrational, is ergodic. This transformation has even stronger properties of unique ergodicity, minimality, and equidistribution. By contrast, if $\theta = p/q$ is rational (in lowest terms) then T is periodic, with period q, and thus cannot be ergodic: for any interval I of length a, $0 < a < 1/q$, its orbit under T is a T-invariant mod 0 set that is a union of q intervals of length a, hence it has measure qa strictly between 0 and 1.
- Let G be a compact abelian group, μ the normalized Haar measure, and T a group automorphism of G. Let G^* be the Pontryagin dual group, consisting of the continuous characters of G, and T^* be the corresponding adjoint automorphism of G^*. The automorphism T is ergodic if and only if the equality $(T^*)^n(\chi)=\chi$ is possible only when $n = 0$ or χ is the trivial character of G. In particular, if G is the n-dimensional torus and the automorphism T is represented by an integral matrix A then T is ergodic if and only if no eigenvalue of A is a root of unity.
- A Bernoulli shift is ergodic. More generally, ergodicity of the shift transformation associated with a sequence of i.i.d. random variables and some more general stationary processes follows from Kolmogorov's zero-one law.
- Ergodicity of a continuous dynamical system means that its trajectories "spread around" the phase space. A system with a compact phase space which has a non-constant first integral cannot be ergodic. This applies, in particular, to Hamiltonian systems with a first integral I functionally independent from the Hamilton function H and a compact level set $X = \{(p,q): H(p,q)=E\}$ of constant energy. Liouville's theorem implies the existence of a finite invariant measure on X, but the dynamics of the system is constrained to the level sets of I on X, hence the system possesses invariant sets of positive but less than full measure. A property of continuous dynamical systems that is the opposite of ergodicity is complete integrability.

Ergodic theorems

Let $T: X \to X$ be a measure-preserving transformation on a measure space (X, Σ, μ). One may then consider the "time average" of a μ-integrable function f, i.e. $f \in L^1(\mu)$. The "time average" is defined as the average (if it exists) over iterations of T starting from some initial point x.

$$\hat{f}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x\right).$$

If $\mu(X)$ is finite and nonzero, we can consider the "space average" or "phase average" of f, defined as

$$\bar{f} = \frac{1}{\mu(X)} \int f \, d\mu.$$ (For a probability space, $\mu(X) = 1$)

In general the time average and space average may be different. But if the transformation is ergodic, and the measure is invariant, then the time average is equal to the space average almost everywhere. This is the celebrated ergodic theorem, in an abstract form due to George David Birkhoff. (Actually, Birkhoff's paper considers not the abstract general case but only the case of dynamical systems arising from differential equations on a smooth manifold.) The equidistribution theorem is a special case of the ergodic theorem, dealing specifically with the distribution of probabilities on the unit interval.

More precisely, the **pointwise** or **strong ergodic theorem** states that the limit in the definition of the time average of *f* exists for almost every *x* and that the (almost everywhere defined) limit function $\hat{f}$ is integrable:

$$\hat{f} \in L^1(\mu).$$

Furthermore, $\hat{f}$ is *T*-invariant, that is to say

$$\hat{f} \circ T = \hat{f}$$

holds almost everywhere, and if $\mu(X)$ is finite, then the normalization is the same:

$$\int \hat{f}\, d\mu = \int f\, d\mu.$$

In particular, if *T* is ergodic, then $\hat{f}$ must be a constant (almost everywhere), and so one has that

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almost everywhere. Joining the first to the last claim and assuming that $\mu(X)$ is finite and nonzero, one has that

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As an example, assume that the measure space (X, Σ, μ) models the particles of a gas as above, and let *f*(*x*) denotes the velocity of the particle at position *x*. Then the pointwise ergodic theorems says that the average velocity of all particles at some given time is equal to the average velocity of one particle over time.

Probabilistic formulation: Birkhoff-Khinchin theorem

Birkhoff-Khinchin theorem. Let f be measurable, $E(|f|) < +\infty$, and T be a measure-preserving operator. Then

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where $E(f|\mathcal{C})$ is the conditional expectation given the σ -algebra $\mathcal{C}$ of invariant sets of T .
Corollary (**Pointwise ergodic theorem**) In particular, if T is also ergodic, then $\mathcal{C}$ is the trivial σ -algebra, and thus

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x\right) = E(f) \text{ a.s.}$$

Mean ergodic theorem

Another form of the ergodic theorem, **von Neumann's mean ergodic theorem**, holds in Hilbert spaces.[1]

Let U be a unitary operator on a → Hilbert space H . Let P be the orthogonal projection onto $\{\psi \in H | U\psi = \psi\} = \mathrm{Ker}(\mathrm{id} - U)$.

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converges to *P* in the strong operator topology.

This theorem specializes to the case in which the Hilbert space *H* consists of L^2 functions on a measure space and *U* is an operator of the form

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Remark: Some intuition for the mean ergodic theorem can be developed by considering the case where complex numbers of unit length are regarded as unitary transformations on the complex plane (by left multiplication). If we pick a single complex number of unit length (which we think of as U), it is intuitive that its powers will fill up the circle. Since the circle is symmetric around 0, it makes sense that the averages of the powers of U will converge to 0. Also, 0 is the only fixed point of U , and so the the projection onto the space of fixed points must be the zero operator (which agrees with the limit just described).

Sojourn time

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$$\frac{R_1 + \cdots + R_n}{n} \to \frac{\mu(X)}{\mu(A)} \quad \text{(almost surely)}$$

(See almost surely.) That is, the smaller *A* is, the longer it takes to return to it.

Ergodic flows on manifolds

The ergodicity of the geodesic flow on compact Riemann surfaces of variable negative curvature and on compact manifolds of constant negative curvature of any dimension was proved by Eberhard Hopf in 1939, although special cases had been studied earlier: see for example, Hadamard's billiards (1898) and Artin billiard (1924). The relation between geodesic flows on Riemann surfaces and one-parameter subgroups on SL(2,**R**) was described in 1952 by S. V. Fomin and I. M. Gelfand. The article on Anosov flows provides an example of ergodic flows on SL(2,**R**) and on Riemann surfaces of negative curvature. Much of the development described there generalizes to hyperbolic manifolds, since they can be viewed as quotients of the hyperbolic space by the action of a lattice in the semisimple Lie group SO(n,1). Ergodicity of the geodesic flow on Riemannian symmetric spaces was demonstrated by F. I. Mautner in 1957. In 1967 D. V. Anosov and Ya. G. Sinai proved ergodicity of the geodesic flow on compact

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See also

- Chaos theory
- → Dynamical systems theory
- Ergodic hypothesis
- Ergodic process
- Functional analysis
- Maximal ergodic theorem
- Poincaré recurrence theorem
- Statistical mechanics
- Markov chain

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Measure-preserving dynamical system

In mathematics, a **measure-preserving dynamical system** is an object of study in the abstract formulation of dynamical systems, and → ergodic theory in particular.

Definition

A measure-preserving dynamical system is defined as a probability space and a measure-preserving transformation on it. In more detail, it is a system

$$(X, \mathcal{B}, \mu, T)$$

with the following structure:

- X is a set,
- $\mathcal{B}$ is a σ-algebra over X ,
- $\mu : \mathcal{B} \to [0, 1]$ is a probability measure, so that $\mu(X) = 1$, and
- $T : X \to X$ is a measurable transformation which preserves the measure μ , i. e. each $A \in \mathcal{B}$ satisfies

$$\mu(T^{-1}A) = \mu(A).$$

This definition can be generalized to the case in which T is not a single transformation that is iterated to give the dynamics of the system, but instead is a monoid (or even a group) of transformations $T_s : X \to X$ parametrized by $s \in \mathbb{Z}$ (or $\mathbb{R}$, or $\mathbb{N} \cup \{0\}$, or $[0, +\infty)$), where each transformation T_s satisfies the same requirements as T above. In particular, the transformations obey the rules

- $T_0 = \mathrm{id}_X : X \to X$, the identity function on X ;
- $T_s \circ T_t = T_{t+s}$, whenever all the terms are well-defined;
- $T_s^{-1} = T_{-s}$, whenever all the terms are well-defined.

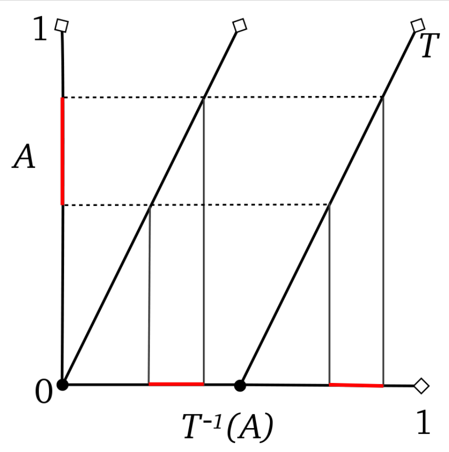
The earlier, simpler case fits into this framework by defining $T_s := T^s$ for $s \in \mathbb{N}$.

The existence of invariant measures for certain maps and Markov processes is established by the Krylov–Bogolyubov theorem.

Examples

Examples include:

- μ could be the normalized angle measure dθ/2π on the unit circle, and T a rotation. See equidistribution theorem;
- the Bernoulli scheme;
- the interval exchange transformation;
- with the definition of an appropriate measure, a subshift of finite type;
- the base flow of a random dynamical system.



Example of a (Lebesgue) measure preserving map: $T\colon [0,1) \to [0,1),$ $x \mapsto 2x \mod 1.$

Homomorphisms

The concept of a homomorphism and an isomorphism may be defined.

Consider two dynamical systems $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$. Then a mapping

$$\phi : X \to Y$$

is a **homomorphism of dynamical systems** if it satisfies the following three properties:

1. The map φ is measurable,
2. For each $B \in \mathcal{B}$, one has $\mu(\phi^{-1}B) = \nu(B)$,
3. For μ-almost all $x \in X$, one has $\phi(Tx) = S(\phi x)$.

The system $(Y, \mathcal{B}, \nu, S)$ is then called a **factor** of $(X, \mathcal{A}, \mu, T)$.

The map φ is an **isomorphism of dynamical systems** if, in addition, there exists another mapping

$$\psi : Y \to X$$

that is also a homomorphism, which satisfies

1. For μ-almost all $x \in X$, one has $x = \psi(\phi x)$
2. For ν-almost all $y \in Y$, one has $y = \phi(\psi y)$.

Generic points

A point $x \in X$ is called a **generic point** if the orbit of the point is distributed uniformly according to the measure.

Symbolic names and generators

Consider a dynamical system $(X, \mathcal{B}, T, \mu)$, and let $Q = \{ Q_1, ..., Q_k \}$ be a partition of X into k measurable pair-wise disjoint pieces. Given a point $x \in X$, clearly x belongs to only one of the Q_i. Similarly, the iterated point $T^n x$ can belong to only one of the parts as well. The **symbolic name** of x, with regards to the partition Q, is the sequence of integers $\{a_n\}$ such that

$$T^n x \in Q_{a_n}.$$

The set of symbolic names with respect to a partition is called the → symbolic dynamics of the dynamical system. A partition Q is called a **generator** or **generating partition** if μ-almost every point x has a unique symbolic name.

Operations on partitions

Given a partition Q = $\{ Q_1, ..., Q_k \}$ and a dynamical system $(X, \mathcal{B}, T, \mu)$, we define T-pullback of Q as

$$T^{-1}Q = \{T^{-1}Q_1, \ldots, T^{-1}Q_k\}.$$

Further, given two partitions $Q = \{ Q_1, ..., Q_k \}$ and $R = \{ R_1, ..., R_k \}$, we define their *refinement* $Q \vee R$ as

$$Q \vee R = \{Q_i \cap R_j \mid i = 1, \ldots, k,\ j = 1, \ldots, m,\ \mu(Q_i \cap R_j) > 0\}.$$

With these two constructs we may define *refinement of an iterated pullback*

$$\begin{aligned} \vee_{n=0}^{N} T^{-n}Q = \{Q_{i_0} \cap T^{-1}Q_{i_1} \cap \cdots \cap T^{-N}Q_{i_N} \\ \mid i_\ell = 1, \ldots, k,\ \ell = 0, \ldots, N, \\ \mu(Q_{i_0} \cap T^{-1}Q_{i_1} \cap \cdots \cap T^{-N}Q_{i_N}) > 0\} \end{aligned}$$

which plays crucial role in the construction of the measure-theoretic entropy of a dynamical system.

Measure-theoretic entropy

The entropy of a partition Q is defined as

$$H(Q) = -\sum_{m=1}^{k} \mu(Q_m) \log \mu(Q_m).$$

The measure-theoretic entropy of a dynamical system $(X, \mathcal{B}, T, \mu)$ with respect to a partition $Q = \{ Q_1, ..., Q_k \}$ is then defined as

$$h_\mu(T,Q) = \lim_{N\rightarrow\infty} \frac{1}{N} H\left(\bigvee_{n=0}^N T^{-n}Q\right).$$

Finally, the **Kolmogorov–Sinai** or **measure-theoretic entropy** of a dynamical system $(X, \mathcal{B}, \mu, T)$ is defined as

$$h_\mu(T) = \sup_Q h_\mu(T,Q).$$

where the supremum is taken over all finite measurable partitions. A theorem of Yakov G. Sinai in 1959 shows that the supremum is actually obtained on partitions that are generators. Thus, for example, the entropy of the Bernoulli process is log 2, since every real number has a unique binary expansion. That is, one may partition the unit interval into the intervals [0, 1/2) and [1/2, 1]. Every real number x is either less than 1/2 or not; and likewise so is the fractional part of $2^n x$.

If the space X is endowed with a metric, then the topological entropy may also be defined.

See also

- Krylov–Bogolyubov theorem on the existence of invariant measures

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Examples

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Periodic orbit

In mathematics, in the study of → dynamical systems, an **orbit** is a collection of points related by the evolution function of the dynamical system. The orbit is a subset of the phase space and the set of all orbits is a partition of the phase space, that is different orbits do not intersect in the phase space. Understanding the properties of orbits by using topological method is one of the objectives of the modern theory of dynamical systems.

For discrete-time dynamical systems the orbits are sequences, for real dynamical systems the orbits are curves and for holomorphic dynamical systems the orbits are Riemann surfaces.

Definition

Given a dynamical system (T, M, Φ) with T a group, M a set and Φ the evolution function

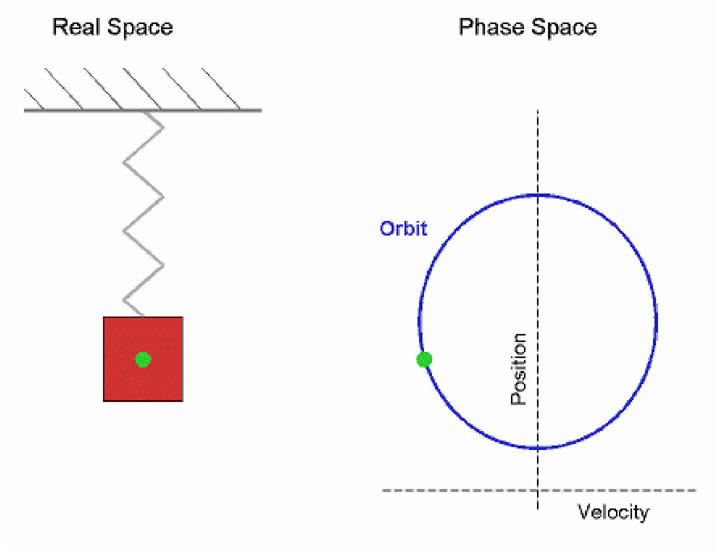


Diagram showing the periodic orbit of a mass-spring system in simple harmonic motion. (Here the velocity and position axes have been reversed from the standard convention in order to align the two diagrams)

$\Phi : U \to M$ where $U \subset T \times M$

we define

$$I(x) := \{t \in T : (t, x) \in U\},$$

then the set

$$\gamma_x := \{\Phi(t, x) : t \in I(x)\}$$

is called **orbit** through x. An orbit which consists of a single point is called **constant orbit**. A non-constant orbit is called **closed** or **periodic** if there exists a t in T so that

$$\Phi(t, x) = x$$

for every point x on the orbit.

Real dynamical system

Given a real dynamical system (R, M, Φ), $I(x)$ is an open interval in the real numbers, that is $I(x) =]t_x^-, t_x^+[$. For any x in M

$$\gamma_x^+ := \{\Phi(t,x) : t \in]0, t_x^+[\}$$

is called **positive semi-orbit** through x and

$$\gamma_x^- := \{\Phi(t,x) : t \in]t_x^-, 0[\}$$

is called **negative semi-orbit** through x.

Discrete time dynamical system

For discrete time dynamical system :

forward orbit of x is a set :

$$\gamma_x^+ \overset{\mathrm{def}}{=} \{\Phi(t,x) : t \geq 0\}$$

backward orbit of x is a set :

$$\gamma_x^- \overset{\mathrm{def}}{=} \{\Phi(-t,x) : t \geq 0\}$$

and **orbit** of x is a set :

$$\gamma_x \overset{\mathrm{def}}{=} \gamma_x^- \cup \gamma_x^+$$

where :

- $\Phi$is an evolution function $\Phi : X \to X$ which is here an iterated function,
- set X is **dynamical space**,
- t is number of iteration, which is natural number and $t \cup T$
- x is initial state of system and $x \cup X$

Usually different notation is used :

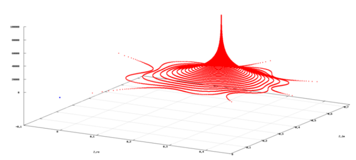
- $\Phi(t,x)$is noted as $\Phi^t(x)$
- $x_t = \Phi^t(x)$with $x_0$is a x from above notation.

Notes

It is often the case that the evolution function can be understood to compose the elements of a group, in which case the group-theoretic orbits of the group action are the same thing as the dynamical orbits.

Examples

- The orbit of an equilibrium point is a constant orbit



Critical orbit of discrete dynamical system based on complex quadratic polynomial. It tends to weakly attracting fixed point with multiplier=0.99993612384259

Stability of orbits

A basic classification of orbits is

- constant orbits or fixed points
- periodic orbits
- non-constant and non-periodic orbits

An orbit can fail to be closed in two interesting ways. It could be an **asymptotically periodic** orbit if it converges to a periodic orbit. Such orbits are not closed because they never truly repeat, but they become arbitrarily close to a repeating orbit. An orbit

can also be chaotic. These orbits come arbitrarily close to the initial point, but fail to ever converge to a periodic orbit. They exhibit sensitive dependence on initial conditions, meaning that small differences in the initial value will cause large differences in future points of the orbit.

There are other properties of orbits that allow for different classifications. An orbit can be hyperbolic if nearby points approach or diverge from the orbit exponentially fast.

See also

- Wandering set
- Phase space method
- Cobweb plot or Verhulst diagram
- Periodic points of complex quadratic mappings and multiplier of orbit

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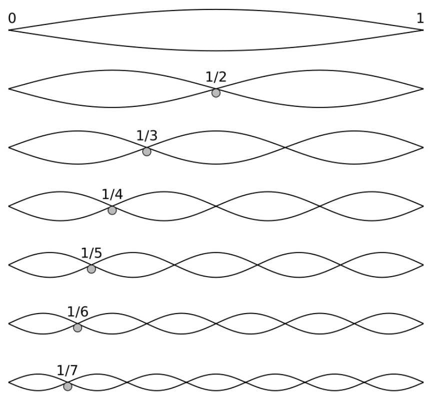
Hilbert space



Hilbert spaces can be used to study the harmonics of vibrating strings.

The mathematical concept of a **Hilbert space**, named after David Hilbert, generalizes the notion of Euclidean space. It extends the methods of vector algebra and calculus from the two-dimensional Euclidean plane and three-dimensional space to spaces with any finite or infinite number of dimensions. A Hilbert space is an abstract vector space possessing the structure of an inner product that allows length and angle to be measured. Hilbert spaces are in addition required to be *complete*, a property that stipulates the existence of enough limits in the space to allow the techniques of calculus to be used.

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. The earliest Hilbert spaces were studied from this point of view in the first decade of the 20th century by David Hilbert, Erhard Schmidt, and Frigyes Riesz. They are indispensable tools in the theories of partial differential equations, quantum mechanics, Fourier analysis which includes applications to signal processing, and → ergodic theory which forms the mathematical underpinning of the study of thermodynamics. John von Neumann coined the term "Hilbert space" for the abstract concept underlying many of these diverse applications. The success of Hilbert space methods ushered in a very fruitful era for functional analysis. Apart from the classical Euclidean spaces, examples of Hilbert spaces include spaces of square-integrable functions, spaces of sequences, Sobolev spaces consisting of generalized functions, and Hardy spaces of holomorphic functions.

Geometric intuition plays an important role in many aspects of Hilbert space theory. An analog of the Pythagorean theorem and parallelogram law hold in a Hilbert space. At a deeper level, perpendicular projection onto a subspace (the analog of "dropping the altitude" of a triangle) plays a significant role in optimization problems and other aspects of the theory. An element of a Hilbert space can be uniquely specified by its coordinates with respect to a set of coordinate axes (an orthonormal basis), in analogy with Cartesian coordinates in the plane. When that set of axes is countably infinite, this means that the Hilbert space can also usefully be thought of in terms of infinite sequences

that are square-summable. Linear operators on a Hilbert space are likewise fairly concrete objects: in good cases, they are simply transformations that stretch the space by different factors in mutually perpendicular directions in a sense that is made precise by the study of their spectral theory.

Definition and illustration

First example: Euclidean space

One of the most familiar examples of a Hilbert space is the Euclidean space consisting of three-dimensional vectors, denoted by $\mathbf{R}^3$, and equipped with the dot product. The dot product takes two vectors **x** and **y**, and produces a real number **x**·**y**. If **x** and **y** are represented in Cartesian coordinates, then the dot product is defined by

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

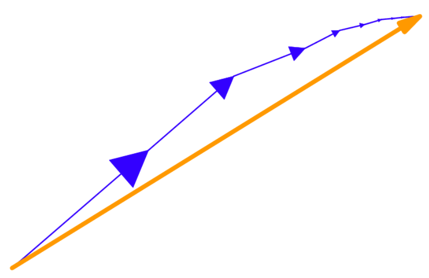
The dot product satisfies the properties:

1. It is symmetric in **x** and **y**: **x**·**y** = **y**·**x**.
2. It is linear in its first argument: $(a\mathbf{x}_1 + b\mathbf{x}_2)\cdot\mathbf{y} = a\mathbf{x}_1\cdot\mathbf{y} + b\mathbf{x}_2\cdot\mathbf{y}$ for any scalars *a*, *b*, and vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and **y**.
3. It is positive definite: for all vectors **x**, **x**·**x** ≥ 0 with equality if and only if **x** = 0.

An operation on pairs of vectors that, like the dot product, satisfies these three properties is known as a (real) inner product. A vector space equipped with such an inner product is known as a (real) inner product space. Every finite-dimensional inner product space is also a Hilbert space. The basic feature of the dot product that connects it with Euclidean geometry is that it is related to both the length (or norm) of a vector, denoted ||**x**||, and to the angle θ between two vectors **x** and **y** by means of the formula

$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\,\cos\theta.$$

Multivariable calculus in Euclidean space relies on the ability to compute limits, and to have useful criteria for concluding that limits exist. A mathematical series



Completeness means that if a particle moves along the broken path (in blue) travelling a finite total distance, then the particle has a well-defined net displacement (in yellow).

$$\sum_{n=0}^{\infty} \mathbf{x}_n$$

consisting of vectors in $\mathbf{R}^3$ is absolutely convergent provided that the sum of the lengths converges as an ordinary series of real numbers:[1]

$$\sum_{k=0}^{\infty} \|\mathbf{x}_k\| < \infty.$$

Just as with a series of scalars, a series of vectors that converges absolutely also converges to some limit vector **L** in the Euclidean space, in the sense that

$$\left\|\mathbf{L} - \sum_{k=0}^{N} \mathbf{x}_k\right\| \to 0 \quad \text{as } N \to \infty.$$

This property expresses the *completeness* of Euclidean space: that a series which converges absolutely also converges in the ordinary sense.

Definition

A **Hilbert space** H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.[2] To say that H is a complex inner product space means that H is a complex vector space on which there is an inner product $\langle x,y\rangle$ associating a complex number to each pair of elements x,y of H, that satisfies the properties:

- $\langle y,x\rangle$ is the complex conjugate of $\langle x,y\rangle$:

$$\langle y,x\rangle = \overline{\langle x,y\rangle}.$$

- $\langle x,y\rangle$ is linear in its first argument. For all complex numbers a and b,

$$\langle ax_1 + bx_2, y\rangle = a\langle x_1, y\rangle + b\langle x_2, y\rangle.$$

- $\langle x,y\rangle$ is positive definite:

$$\langle x,x\rangle \geq 0$$

  where the case of equality holds precisely when $x = 0$.

A real inner product space is defined in the same way, except that H is a real vector space and the inner product takes real values.

The norm defined by the inner product $\langle \bullet,\bullet\rangle$ is the real-valued function

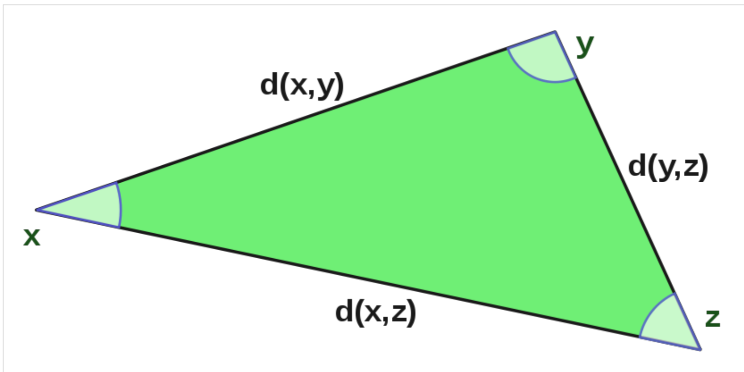
$$\|x\| = \sqrt{\langle x,x\rangle},$$

and the distance between two points x,y in H is defined in terms of the norm by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}.$$

That this function is a distance function means (1) that it is symmetric in x and y, (2) that the distance between x and itself is zero, and otherwise the distance between x and y must be positive, and (3) that the triangle inequality holds, meaning that the length of one leg of a triangle xyz cannot exceed the sum of the lengths of the other two legs:

$$d(x,z) \leq d(x,y) + d(y,z).$$



This last property is ultimately a consequence of the more fundamental Cauchy–Schwarz inequality, which asserts

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|$$

with equality if and only if x and y are parallel.

Relative to a distance function defined in this way, any inner product space is a metric space, and sometimes is known as a **pre-Hilbert space**.[3] A pre-Hilbert space is a Hilbert space if in addition it is complete. Completeness is expressed using a form of the Cauchy criterion for sequences in H: a pre-Hilbert space H is complete if every Cauchy sequence converges with respect to this norm to an element in the space. Completeness can be characterized by the following equivalent condition: if a series of vectors $\sum_{k=0}^{\infty} u_k$ converges absolutely in the sense that

$$\sum_{k=0}^{\infty} \|u_k\| < \infty,$$

then the series converges in H, in the sense that the partial sums converge to an element of H.

As a complete normed space, Hilbert spaces are by definition also Banach spaces. As such they are topological vector spaces, in which topological notions like the openness and closedness of subsets are well-defined. Of special importance is the notion of a closed linear subspace of a Hilbert space which, with the inner product induced by restriction, is also complete (being a closed set in a complete metric space) and therefore a Hilbert space in its own right.

Second example: sequence spaces

The sequence space ℓ^2 consists of all infinite sequences **z** = (z_1,z_2,...) of complex numbers such that the series

$$\sum_{n=1}^{\infty} |z_n|^2$$

converges. The inner product on ℓ^2 is defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n},$$

with the latter series converging as a consequence of the Cauchy–Schwarz inequality.

Completeness of the space holds provided that whenever a series of elements from ℓ^2 converges absolutely (in norm), then it converges to an element of ℓ^2. The proof is basic in mathematical analysis, and permits mathematical series of elements of the space to be manipulated with the same ease as series of complex numbers (or vectors in a finite-dimensional Euclidean space).[4]

History

Prior to the development of Hilbert spaces, other generalizations of Euclidean spaces were known to mathematicians and physicists. In particular, the idea of an abstract linear space had gained some traction towards the end of the 19th century:[5] this is a space whose elements can be added together and multiplied by scalars (such as real or complex numbers) without necessarily identifying these elements with "geometric" vectors, such as position and momentum vectors in physical systems. Other objects studied by mathematicians at the turn of the 20th century, in particular spaces of sequences (including series) and spaces of functions,[6] can naturally be thought of as linear spaces. Functions, for instance, can be added together or multiplied by constant scalars, and these operations obey the algebraic laws satisfied by addition and scalar multiplication of spatial vectors.

David Hilbert

In the first decade of the 20th century, parallel developments led to the introduction of Hilbert spaces. The first of these was the observation, which arose during David Hilbert and Erhard Schmidt's study of integral equations,[7] that two square-integrable real-valued functions f and g on an interval [a,b] have an *inner product*

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

which has many of the familiar properties of the Euclidean dot product. In particular, the idea of an orthogonal family of functions has meaning. Schmidt exploited the similarity of this inner product with the usual dot product to prove an analog of the spectral decomposition for an operator of the form

$$f(x) \mapsto \int_a^b K(x,y)f(y)\,dy$$

where K is a continuous function symmetric in x and y. The resulting eigenfunction expansion expresses the function K as a series of the form

$$K(x,y) = \sum_n \lambda_n \varphi_n(x)\varphi_n(y)$$

where the functions φ_n are orthogonal in the sense that $\langle \varphi_n, \varphi_m \rangle = 0$ for all $n \neq m$. However, there are eigenfunction expansions which fail to converge in a suitable sense to a square-integrable function: the missing ingredient, which ensures convergence, is completeness.[8]

The second development was the Lebesgue integral, an alternative to the Riemann integral introduced by Henri Lebesgue in 1904.[9] The Lebesgue integral made it possible to integrate a much broader class of functions. In 1907, Frigyes Riesz and Ernst Sigismund Fischer independently proved that the space L^2 of square Lebesgue-integrable functions is a complete metric space.[10] As a consequence of the interplay between geometry and completeness, the 19th century results of Joseph Fourier, Friedrich Bessel and Marc-Antoine Parseval on trigonometric series easily carried over to these more general spaces, resulting in a geometrical and analytical apparatus now usually known as the Riesz-Fischer theorem.[11]

Further basic results were proved in the early 20th century. For example, the Riesz representation theorem was independently established by Maurice Fréchet and Frigyes Riesz in 1907.[12] John von Neumann coined the term *abstract Hilbert space* in his work on unbounded Hermitian operators.[13] Although other mathematicians such as Hermann Weyl and Norbert Wiener had already studied particular Hilbert spaces in great detail, often from a physically-motivated point of view, von Neumann gave the first complete and axiomatic treatment of them.[14] Von Neumann later used them in his seminal work on the foundations of quantum mechanics,[15] and in his continued work with Eugene Wigner. The name "Hilbert space" was soon adopted by others, for example by Hermann Weyl in his book on quantum mechanics and the theory of groups.[16]

The significance of the concept of a Hilbert space was underlined with the realization that it offers one of the best mathematical formulations of quantum mechanics.[17] In short, the states of a quantum mechanical system are vectors in a certain Hilbert space, the observables are hermitian operators on that space, the symmetries of the system are unitary operators, and measurements are orthogonal projections. The relation between quantum mechanical symmetries and unitary operators provided an impetus for the development of the unitary representation theory of groups, initiated in the 1928 work of Hermann Weyl.[16] On the other hand, in the early 1930s it became clear that certain properties of classical dynamical systems can be analyzed using Hilbert space techniques in the framework of → ergodic theory.[18]

The algebra of observables in quantum mechanics is naturally an algebra of operators defined on a Hilbert space, according to Werner Heisenberg's matrix mechanics formulation of quantum theory. Von Neumann began investigating operator algebras in the 1930s, as rings of operators on a Hilbert space. The kind of algebras studied by von Neumann and his contemporaries are now known as von Neumann algebras. In the 1940s, Israel Gelfand, Mark Naimark and Irving Segal gave a definition of a kind of operator algebras called C^*-algebras that on the one hand made no reference to an underlying Hilbert space, and on the other extrapolated many of the useful features of the operator algebras that had previously been studied. The spectral theorem for self-adjoint operators in particular that underlay much of the existing Hilbert space theory was generalized to C^*-algebras. These techniques are now basic in abstract harmonic analysis and representation theory.

Examples

Lebesgue spaces

Lebesgue spaces are function spaces associated to measure spaces (X, M, μ), where X is a set, M is a σ-algebra of subsets of X, and μ is a countably additive measure on M. Let $L^2(X,\mu)$ be the space of those complex-valued measurable functions on X for which the Lebesgue integral of the square of the absolute value of the function is finite, and where functions are identified if and only if they differ only on a set of measure 0.

The inner product of functions f and g in $L^2(X,\mu)$ is then defined as

$$\langle f, g\rangle = \int_X f(t)\overline{g(t)}\ d\mu(t).$$

This integral exists, and the resulting space is complete.[19] The Lebesgue integral is essential to ensure completeness: on domains of real numbers, for instance, not enough functions are Riemann integrable.[20]

Sobolev spaces

Sobolev spaces, denoted by H^s or $W^{s,\,2}$, are Hilbert spaces. These are a special kind of function space in which differentiation may be performed, but which (unlike other Banach spaces such as the Hölder spaces) support the structure of an inner product. Because differentiation is permitted, Sobolev spaces are a convenient setting for the theory of partial differential equations.[21] They also form the basis of the theory of direct methods in the calculus of variations.[22]

For s a non-negative integer and $\Omega \subset \mathbf{R}^n$, the Sobolev space $H^s(\Omega)$ contains L^2 functions whose weak derivatives of order up to s are also L^2. The inner product in $H^s(\Omega)$ is

$$\langle f, g\rangle = \int_\Omega f(x)\bar{g}(x)\,dx + \int_\Omega Df\cdot D\bar{g}(x) + \ldots + \int_\Omega D^s f(x)\cdot D^s\bar{g}(x)\,dx$$

where the dot indicates the dot product in the Euclidean space of partial derivatives of each order. Sobolev spaces can also be defined when s is not an integer.

Sobolev spaces are also studied from the point of view of spectral theory, relying more specifically on the Hilbert space structure. If Ω is a suitable domain, then one can define the Sobolev space $H^s(\Omega)$ as the space of Bessel potentials;[23] roughly,

$$H^s(\Omega) = \{(1-\Delta)^{-s/2} f \,|\, f \in L^2(\Omega)\}.$$

Here Δ is the Laplacian and $(1 - \Delta)^{-s/2}$ is understood in terms of the spectral mapping theorem. Apart from providing a workable definition of Sobolev spaces for non-integer s, this definition also has particularly desirable properties under the Fourier transform that make it ideal for the study of pseudodifferential operators. Using these methods on a compact Riemannian manifold, one can obtain for instance the Hodge decomposition which is the basis of Hodge theory.[24]

Spaces of holomorphic functions

Hardy spaces

The Hardy spaces are function spaces, arising in complex analysis and harmonic analysis, whose elements are certain holomorphic functions in a complex domain.[25] Let U denote the unit disc in the complex plane. Then the Hardy space $H^2(U)$ is defined to be the space of holomorphic functions f on U such that the means

$$M_r(f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta$$

remain bounded for $r < 1$. The norm on this Hardy space is defined by

$$\|f\|_2 = \lim_{r\to 1}\sqrt{M_r(f)}.$$

Hardy spaces in the disc are related to Fourier series. A function f is in $H^2(U)$ if and only if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Thus $H^2(U)$ consists of those functions which are L^2 on the circle, and whose negative frequency Fourier coefficients vanish.

Bergman spaces

The Bergman spaces are another family of Hilbert spaces of holomorphic functions.[26] Let D be a bounded open set in the complex plane (or a higher dimensional complex space) and let $L^{2,h}(D)$ be the space of holomorphic functions f in D that are also in $L^2(D)$ in the sense that

$$\|f\|^2 = \int_D |f(z)|^2 \, d\mu(z) < \infty,$$

where the integral is taken with respect to the Lebesgue measure in D. Clearly $L^{2,h}(D)$ is a subspace of $L^2(D)$; in fact, it is a closed subspace, and so a Hilbert space in its own right. This is a consequence of the estimate, valid on compact subsets K of D, that

$$\sup_{z\in K} |f(z)| \le C_K \|f\|_2,$$

which in turn follows from Cauchy's integral formula. Thus convergence of a sequence of holomorphic functions in $L^2(D)$ implies also compact convergence, and so the limit function is also holomorphic. Another consequence of this inequality is that the linear functional that evaluates a function f at a point of D is actually continuous on $L^{2,h}(D)$. The Riesz representation theorem implies that the evaluation functional can be represented as an element of $L^{2,h}(D)$. Thus, for every $z \in D$, there is a function $\eta_z \in L^{2,h}(D)$ such that

$$f(z) = \int_D f(\zeta)\overline{\eta_z(\zeta)} \, d\mu(\zeta)$$

for all $f \in L^{2,h}(D)$. The integrand

$$K(\zeta, z) = \overline{\eta_z(\zeta)}$$

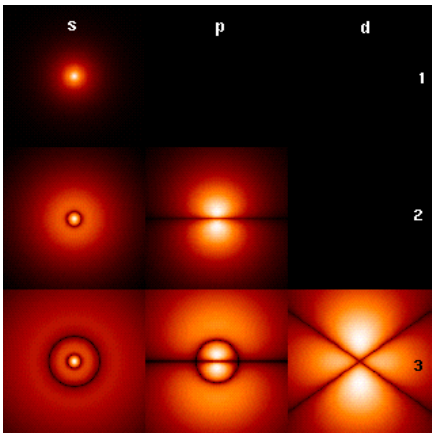
is known as the Bergman kernel of D. This integral kernel satisfies a reproducing property

$$f(z) = \int_D f(\zeta)K(\zeta, z) \, d\mu(\zeta).$$

A Bergman space is an example of a reproducing kernel Hilbert space, which is a Hilbert space of functions along with a kernel $K(\zeta,z)$ that verifies a reproducing property analogous to this one. The Hardy space $H^2(D)$ also admits a reproducing kernel, known as the Szegő kernel.[27] Reproducing kernels are common in other areas of mathematics as well. For instance, in harmonic analysis the Poisson kernel is a reproducing kernel for the Hilbert space of square-integrable harmonic functions in the unit ball. That the latter is a Hilbert space at all is a consequence of the mean value theorem for harmonic functions.

Applications

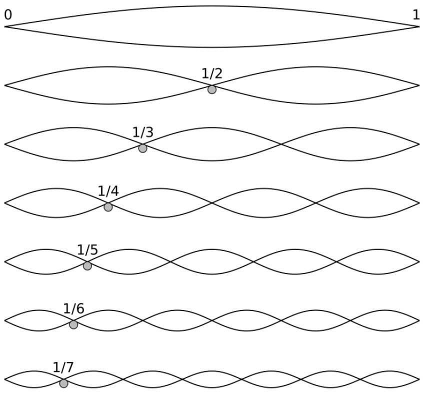
Many of the applications of Hilbert spaces exploit the fact that Hilbert spaces support generalizations of simple geometric concepts like projection and change of basis from their usual finite dimensional setting. In particular, the spectral theory of continuous self-adjoint linear operators on a Hilbert space generalizes the usual spectral decomposition of a matrix, and this often plays a major role in applications of the theory to other areas of mathematics and physics.



The orbitals of an electron in a hydrogen atom are eigenfunctions of the energy.

Sturm–Liouville theory

In the theory of ordinary differential equations, spectral methods on a suitable Hilbert space are used to study the behavior of eigenvalues and eigenfunctions of differential equations. For example, the Sturm–Liouville problem arises in the study of the harmonics of waves in a violin string or a drum, and is a central problem in ordinary differential equations.[28] The problem is a differential equation of the form



The overtones of a vibrating string. These are eigenfunctions of an associated Sturm–Liouville problem. The eigenvalues 1,1/2,1/3,… form the (musical) harmonic series.

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=\lambda w(x)y$$

for an unknown function y on an interval $[a,b]$, satisfying general homogeneous Robin boundary conditions

$$\begin{cases}\alpha y(a)+\alpha' y'(a)=0\\ \beta y(b)+\beta' y'(b)=0.\end{cases}$$

The functions p, q, and w are given in advance, and the problem is to find the function y and constants λ for which the equation has a solution. The problem only has solutions for certain values of λ, called eigenvalues of the system, and this is a consequence of the spectral theorem for compact operators applied to the integral operator defined by the Green's function for the system. Furthermore, another consequence of this general result is that the eigenvalues λ of the system can be arranged in an increasing sequence tending to infinity.[29]

Partial differential equations

Hilbert spaces form a basic tool in the study of partial differential equations.[21] For many classes of partial differential equations, such as linear elliptic equations, it is possible to consider a generalized solution (known as a weak solution) by enlarging the class of functions. Many weak formulation involve the class of Sobolev functions, which is a Hilbert space. A suitable weak formulation reduces to a geometrical problem the analytic problem of finding a solution or, often what is more important, showing that a solution exists and is unique for given boundary data. For linear elliptic equations, one geometrical result that ensures unique solvability for a large class of problems is the Lax–Milgram theorem. This strategy forms the rudiment of the Galerkin method (a finite element method) for numerical solution of partial differential equations.[30]

A typical example is the Poisson equation $-\Delta u = g$ with Dirichlet boundary conditions in a bounded domain Ω in $\mathbf{R}^2$. The weak formulation consists of finding a function *u* such that, for all continuously differentiable functions *v* in Ω vanishing on the boundary:

$$\int_\Omega \nabla u\cdot\nabla v = \int_\Omega gv.$$

This can be recast in terms of the Hilbert space $H^1_0(\Omega)$ consisting of functions *u* such that *u*, along with its weak partial derivatives, are square integrable on Ω, and which vanish on the boundary. The question then reduces to finding *u* in this space such that for all *v* in this space

$$a(u,v) = b(v)$$

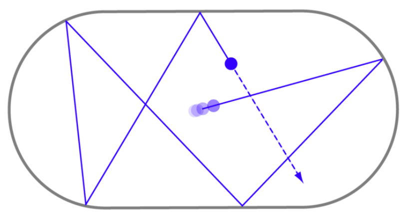
where *a* is a continuous bilinear form, and *b* is a continuous linear functional, given respectively by

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v,\quad b(v) = \int_\Omega gv.$$

Since the Poisson equation is elliptic, it follows from Poincaré's inequality that the bilinear form *a* is coercive. The Lax-Milgram theorem then ensures the existence and uniqueness of solutions of this equation.

Hilbert spaces allow for many elliptic partial differential equations to be formulated in a similar way, and the Lax-Milgram theorem is then a basic tool in their analysis. With suitable modifications, similar techniques can be applied to parabolic partial differential equations and certain hyperbolic partial differential equations.

Ergodic theory



The path of a billiard ball in the Bunimovich stadium is described by an ergodic → dynamical system.

The field of → ergodic theory is the study of the long-term behavior of chaotic → dynamical systems. The protypical case of a field to which ergodic theory is applicable is that of thermodynamics in which, although the microscopic state of a system is extremely complicated—it is impossible to understand the ensemble of individual collisions between particles of matter—the average behavior over sufficiently long time intervals is tractable. The laws of thermodynamics are assertions about such average behavior. In particular, one formulation of the zeroth law of thermodynamics asserts that over sufficiently long timescales, the only functionally independent measurement that one can make of a thermodynamic system in equilibrium is its total energy, in the form of temperature.

An ergodic dynamical system is one for which, apart from the energy—measured by the Hamiltonian—there are no other functionally independent conserved quantities on the phase space. More explicitly, suppose that the energy *E* is fixed, and let Ω_E be the subset of the phase space consisting of all states of energy *E* (an energy surface), and let T_t denote the evolution operator on the phase space. The dynamical system is ergodic if there are no continuous non-constant functions on Ω_E such that

$$f(T_t w) = f(w)$$

for all w on Ω_E and all time t. Liouville's theorem implies that there exists a measure μ on the energy surface that is invariant under the time translation. As a result, time translation is a unitary transformation of the Hilbert space $L^2(\Omega_E,\mu)$ consisting of square-integrable functions on the energy surface Ω_E with respect to the inner product

$$\langle f, g\rangle_{L^2(\Omega_E,\mu)} = \int_E f\bar{g}\,d\mu.$$

The von Neumann mean ergodic theorem[31] states the following:

- If U_t is a (strongly continuous) one-parameter semigroup of unitary operators on a Hilbert space H, and P is the orthogonal projection onto the space of common fixed points of U_t, $\{x\in H \mid U_t x = x \text{ for all } t > 0\}$, then

$$Px = \lim_{T\to\infty} \frac{1}{T}\int_0^T U_t x\,dt.$$

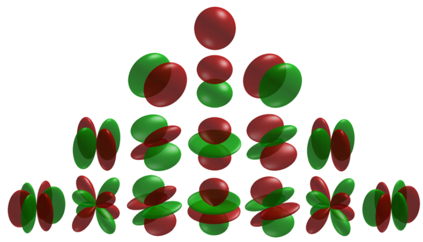
For an ergodic system, the fixed set of the time evolution consists only of the constant functions, so the ergodic theorem implies the following:[32] for any function $f \in L^2(\Omega_E,\mu)$,

$$L^2\!-\!\lim_{T\to\infty}\frac{1}{T}\int_0^T f(T_t w)\,dt = \int_{\Omega_E} f(y)\,d\mu(y).$$

That is, the long time average of an observable f is equal to its expectation value over an energy surface.

Fourier analysis

One of the basic goals of Fourier analysis is to decompose a function into a (possibly infinite) linear combination of given basis functions: the associated Fourier series. The classical Fourier series associated to a function f defined on the interval [0,1] is a series of the form



Spherical harmonics, an orthonormal basis for the Hilbert space of square-integrable functions on the sphere, shown graphed along the radial direction

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\theta}$$

where

$$a_n = \int_0^1 f(\theta)e^{-2\pi i n\theta}\,d\theta.$$

A significant problem in classical Fourier series asks in what sense the Fourier series converges, if at all, to the function f.

Hilbert space methods provide one possible answer to this question.[33] The functions $e_n(\theta) = e^{2\pi i n\theta}$ form an orthogonal basis of the Hilbert space $L^2([0,1])$. Consequently, any square-integrable function can be expressed as a series

$$f(\theta) = \sum_n a_n e_n(\theta), \quad a_n = \langle f, e_n\rangle$$

and, moreover, this series converges in the Hilbert space sense (that is, in the L^2 mean).

The problem can also be studied from the abstract point of view: every Hilbert space has an orthonormal basis, and every element of the Hilbert space can be written in a unique way as a sum of multiples of these basis elements. The

coefficients appearing on these basis elements are sometimes known abstractly as the Fourier coefficients of the element of the space.[34] The abstraction is especially useful when one wishes to use different basis functions for a space such as $L^2([0,1])$. In many circumstances, it is desirable not to decompose a function into trigonometric functions, but rather into orthogonal polynomials or wavelets for instance,[35] and in higher dimensions into spherical harmonics.[36]

In various applications to physical problems, one wishes to decompose a function into physically meaningful eigenfunctions of a differential operator (typically the Laplace operator): this forms the foundation for the spectral study of functions, in reference to the spectrum of the differential operator.[37] A concrete physical application involves the problem of hearing the shape of a drum: given the fundamental modes of vibration that a drumhead is capable of producing, can one infer the shape of the drum itself?[38] The mathematical formulation of this question involves the Dirichlet eigenvalues of the Laplace equation in the plane, that represent the fundamental modes of vibration in direct analogy with the integers that represent the fundamental modes of vibration of the violin string.

Spectral theory also underlies certain aspects of the Fourier transform of a function. Whereas Fourier analysis decomposes a function defined on a compact set into the discrete spectrum of the Laplacian (which corresponds to the vibrations of a violin string or drum), the Fourier transform of a function is the decomposition of a function defined on all of Euclidean space into its components in the continuous spectrum of the Laplacian. The Fourier transformation is also geometrical, in a sense made precise by the Plancherel theorem, that asserts that it is an isometry of one Hilbert space (the "time domain") with another (the "frequency domain"). This isometry property of the Fourier transformation is a recurring theme in abstract harmonic analysis, as evidenced for instance by the Plancherel theorem for spherical functions occurring in noncommutative harmonic analysis.

Quantum mechanics

In the mathematically rigorous formulation of quantum mechanics, developed by Paul Dirac[39] and John von Neumann[40] , the possible states (more precisely, the pure states) of a quantum mechanical system are represented by unit vectors (called *state vectors*) residing in a complex separable Hilbert space, known as the state space, well defined up to a complex number of norm 1 (the phase factor). In other words, the possible states are points in the projectivization of a Hilbert space, usually called the complex projective space. The exact nature of this Hilbert space is dependent on the system; for example, the position and momentum states for a single non-relativistic spin zero particle is the space of all square-integrable functions, while the states for the spin of a single proton are unit elements of the two-dimensional complex Hilbert space of spinors. Each observable is represented by a self-adjoint linear operator acting on the state space. Each eigenstate of an observable corresponds to an eigenvector of the operator, and the associated eigenvalue corresponds to the value of the observable in that eigenstate.

The time evolution of a quantum state is described by the Schrödinger equation, in which the Hamiltonian, the operator corresponding to the total energy of the system, generates time evolution.

The inner product between two state vectors is a complex number known as a probability amplitude. During an ideal measurement of a quantum mechanical system, the probability that a system collapses from a given initial state to a particular eigenstate is given by the square of the absolute value of the probability amplitudes between the initial and final states. The possible results of a measurement are the eigenvalues of the operator—which explains the choice of self-adjoint operators, for all the eigenvalues must be real. The probability distribution of an observable in a given state can be found by computing the spectral decomposition of the corresponding operator.

For a general system, states are typically not pure, but instead are represented as statistical mixtures of pure states, or mixed states, given by density matrices: self-adjoint operators of trace one on a Hilbert space. Moreover, for general quantum mechanical systems, the effects of a single measurement can influence other parts of a system in a manner that is described instead by a positive operator valued measure. Thus the structure both of the states and observables in the general theory is considerably more complicated than the idealization for pure states.

Heisenberg's uncertainty principle is represented by the statement that the operators corresponding to certain observables do not commute, and gives a specific form that the commutator must have.

Properties

Pythagorean identity

Two vectors u and v in a Hilbert space H are orthogonal when $\langle u, v\rangle = 0$. The notation for this is $u \perp v$. More generally, when S is a subset in H, the notation $u \perp S$ means that u is orthogonal to every element from S.

When u and v are orthogonal, one has

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + 2\,\mathrm{Re}\langle u, v\rangle + \langle v, v\rangle = \|u\|^2 + \|v\|^2.$$

By induction on n, this is extended to any family $u_1,\ldots,u_n$ of n orthogonal vectors,

$$\|u_1 + \cdots + u_n\|^2 = \|u_1\|^2 + \cdots + \|u_n\|^2.$$

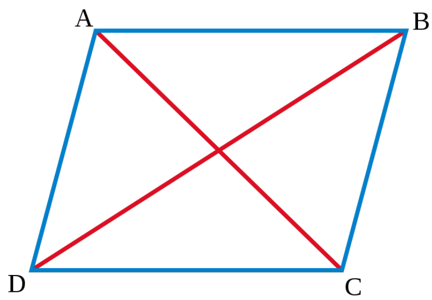
Whereas the Pythagorean identity as stated is valid in any inner product space, completeness is required for the extension of the Pythagorean identity to series. A series $\Sigma\, u_k$ of *orthogonal* vectors converges in H if and only if the series of squares of norms converges, and

$$\Big\|\sum_{k=0}^{\infty} u_k\Big\|^2 = \sum_{k=0}^{\infty} \|u_k\|^2.$$

Furthermore, the sum of a series of orthogonal vectors is independent of the order in which it is taken.

Parallelogram identity and polarization

By definition, every Hilbert space is also a Banach space. Furthermore, in every Hilbert space the following parallelogram identity holds:



Geometrically, the parallelogram identity asserts that $AC^2 + BD^2 = 2(AB^2 + AD^2)$. In words, the sum of the squares of the diagonals is twice the sum of the squares of any two adjacent sides.

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Conversely, every Banach space in which the parallelogram identity holds is a Hilbert space, and the inner product is uniquely determined by the norm by the polarization identity.[41] For real Hilbert spaces, the polarization identity is

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right).$$

For complex Hilbert spaces, it is

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right).$$

The parallelogram law implies that any Hilbert space is a uniformly convex Banach space.[42]

Best approximation

If C is a non-empty closed convex subset of a Hilbert space H and x a point in H, there exists a unique point $y \in C$ which minimizes the distance between x and points in C,[43]

$$y \in C, \quad \|x-y\| = \operatorname{dist}(x, C) = \min\{\|x-z\| : z \in C\}.$$

This is equivalent to saying that there is a point with minimal norm in the translated convex set $D = C - x$. The proof consists in showing that every minimizing sequence $(d_n) \subset D$ is Cauchy (using the parallelogram identity) hence converges (using completeness) to a point in D that has minimal norm. More generally, this holds in any uniformly convex Banach space.[44]

When this result is applied to a closed subspace F of H, it can be shown that the point $y \in F$ closest to x is characterized by[45]

$$y \in F, \quad x - y \perp F.$$

This point y is the *orthogonal projection* of x onto F, and the mapping $P_F : x \to y$ is linear (see Orthogonal complements and projections). This result is especially significant in applied mathematics, especially numerical analysis, where it forms the basis of least squares methods.

In particular, when F is not equal to H, one can find a non-zero vector v orthogonal to F (select x not in F and $v = x - y$). A very useful criterion is obtained by applying this observation to the closed subspace F generated by a subset S of H.

> A subset S of H spans a dense vector subspace if (and only if) the vector 0 is the sole vector $v \in H$ orthogonal to S.

Duality

The dual space H^* is the space of all continuous linear functions from the space H into the base field. It carries a natural norm, defined by

$$\|\varphi\| = \sup_{\|x\|=1, x\in H} |\varphi(x)|.$$

This norm satisfies the parallelogram law, and so the dual space is also an inner product space. The dual space is also complete, and so it is a Hilbert space in its own right.

The Riesz representation theorem affords a convenient description of the dual. To every element u of H, there is a unique element φ_u of H^*, defined by

$$\varphi_u(x) = \langle x, u \rangle.$$

The mapping $u \mapsto \varphi_u$ is an antilinear mapping from H to H^*. The Riesz representation theorem states that this mapping is an antilinear isomorphism.[46] Thus to every element φ of the dual H^* there exists one and only one u_φ in H such that

$$\langle x, u_\varphi \rangle = \varphi(x)$$

for all $x \in H$. The inner product on the dual space H^* satisfies

$$\langle \varphi, \psi \rangle = \langle u_\psi, u_\varphi \rangle.$$

The reversal of order on the right-hand side restores linearity in φ from the antilinearity of u_φ.

The representing vector u_φ is obtained in the following way. When $\varphi \neq 0$, the kernel $F = \ker \varphi$ is a closed vector subspace of H, not equal to H, hence there exists a non-zero vector v orthogonal to F. The vector u is a suitable scalar multiple λv of v. The requirement that $\varphi(v) = \langle v, u \rangle$ yields

$$u = \langle v, v \rangle^{-1} \overline{\varphi(v)}\, v.$$

This correspondence $\varphi \leftrightarrow u$ is exploited by the bra-ket notation popular in physics. It is common in physics to assume that the inner product, denoted by $\langle x|y \rangle$, is linear on the right,

$$\langle x|y \rangle = \langle y, x \rangle.$$

The result $\langle x|y\rangle$ can be seen as the action of the linear functional $\langle x|$ (the *bra*) on the vector $|y\rangle$ (the *ket*).

The Riesz representation theorem relies fundamentally not just on the presence of an inner product, but also on the completeness of the space. In fact, the theorem implies that the topological dual of any inner product space can be identified with its completion. An immediate consequence of the Riesz representation theorem is also that a Hilbert space *H* is reflexive, meaning that the natural map from *H* into its double dual space is an isomorphism.

Weakly convergent sequences

In a Hilbert space *H*, a sequence $\{x_n\}$ is weakly convergent to a vector $x \in H$ when

$$\lim_n \langle x_n, v \rangle = \langle x, v \rangle$$

for every $v \in H$.

For example, any orthonormal sequence $\{f_n\}$ converges weakly to 0, as a consequence of Bessel's inequality. Every weakly convergent sequence $\{x_n\}$ is bounded, by the uniform boundedness principle.

Conversely, every bounded sequence in a Hilbert space admits weakly convergent subsequences (Alaoglu's theorem).[47] This fact may be used to prove minimization results for continuous convex functionals, in the same way that the Bolzano-Weierstrass theorem is used for continuous functions on $\mathbf{R}^d$. Among several variants, one simple statement is as follows:[48]

> If $f : H \to \mathbf{R}$ is a convex continuous function such that $|f(x)|$ tends to $+\infty$ when $\|x\|$ tends to ∞, then f admits a minimum at some point $x_0 \in H$.

This fact (and its various generalizations) are fundamental for direct methods in the calculus of variations. Minimization results for convex functionals are also a direct consequence of the slightly more abstract fact that closed bounded convex subsets in a Hilbert space *H* are weakly compact, since *H* is reflexive. The existence of weakly convergent subsequences is a special case of the Eberlein-Šmulian theorem.

Banach space properties

Any general property of Banach spaces continues to hold for Hilbert spaces. The open mapping theorem states that a continuous surjective linear transformation from one Banach space to another is an open mapping meaning that it sends open sets to open sets. A corollary is the bounded inverse theorem, that a continuous and bijective linear function from one Banach space to another is an isomorphism (that is, a continuous linear map whose inverse is also continuous). This theorem is considerably simpler to prove in the case of Hilbert spaces than in general Banach spaces.[49] The open mapping theorem is equivalent to the closed graph theorem, which asserts that a function from one Banach space to another is continuous if and only if its graph is a closed set.[50] In the case of Hilbert spaces, this is basic in the study of unbounded operators (see closed operator).

The (geometrical) Hahn-Banach theorem asserts that a closed convex set can be separated from any point outside it by means of a hyperplane of the Hilbert space. This is an immediate consequence of the best approximation property: if *y* is the element of a closed convex set *F* closest to *x*, then the separating hyperplane is the plane perpendicular to the segment *xy* passing through its midpoint.[51]

Operators on Hilbert spaces

Bounded operators

The continuous linear operators $A : H_1 \to H_2$ from a Hilbert space H_1 to a second Hilbert space H_2 are *bounded* in the sense that they map bounded sets to bounded sets. Conversely, if an operator is bounded, then it is continuous. The space of such bounded linear operators has a norm, the operator norm given by

$$\|A\| = \sup \{ \, \|Ax\| \, : \, \|x\| \leq 1 \, \} \, .$$

The sum and the composite of two bounded linear operators is again bounded and linear. For y in H_2, the map that sends $x \in H_1$ to $\langle Ax, y \rangle$ is linear and continuous, and according to the Riesz representation theorem can therefore be represented in the form

$$\langle x, A^* y \rangle = \langle Ax, y \rangle$$

for some vector A^*y in H_1. This defines another bounded linear operator $A^* : H_2 \to H_1$, the adjoint of A. One can see that $A^{**} = A$.

The set B(H) of all bounded linear operators on H, together with the addition and composition operations, the norm and the adjoint operation, is a C^*-algebra, which is a type of operator algebra.

An element A of B(H) is called *self-adjoint* or *Hermitian* if $A^* = A$. If A is Hermitian and $\langle Ax, x \rangle \geq 0$ for every x, then A is called *non-negative*, written $A \geq 0$; if equality holds only when $x = 0$, then A is called *positive*. The set of self adjoint operators admits a partial order, in which $A \geq B$ if $A - B \geq 0$. If A has the form B^*B for some B, then A is non-negative; if B is invertible, then A is positive. A converse is also true in the sense that, for a non-negative operator A, there exists a unique non-negative square root B such that

$$A = B^2 = B^*B.$$

In a sense made precise by the spectral theorem, self-adjoint operators can usefully be thought of as operators that are "real". An element A of B(H) is called *normal* if $A^*A = A A^*$. Normal operators decompose into the sum of a self-adjoint operators and an imaginary multiple of a self adjoint operator

$$A = \frac{A + A^*}{2} + i\frac{(A - A^*)}{2i}$$

that commute with each other. Normal operators can also usefully be thought of in terms of their real and imaginary parts.

An element U of B(H) is called unitary if U is invertible and its inverse is given by U^*. This can also be expressed by requiring that U be onto and $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all x and y in H. The unitary operators form a group under composition, which is the isometry group of H.

An element of B(H) is compact if it sends bounded sets to relatively compact sets. Equivalently, a bounded operator T is compact if, for any bounded sequence $\{x_k\}$, the sequence $\{Tx_k\}$ has a convergent subsequence. Many integral operators are compact, and in fact define a special class of operators known as Hilbert–Schmidt operators that are especially important in the study of integral equations. Fredholm operators are those which differ from a compact operator by a multiple of the identity, and are equivalently characterized as operators with a finite dimensional kernel and cokernel. The index of a Fredholm operator T is defined by

$$\operatorname{index} T = \dim \ker T - \dim \operatorname{coker} T.$$

The index is homotopy invariant, and plays a deep role in differential geometry via the Atiyah–Singer index theorem.

Unbounded operators

Unbounded operators are also tractable in Hilbert spaces, and have important applications to quantum mechanics.[52] An unbounded operator T on a Hilbert space H is defined to be a linear operator whose domain $D(T)$ is a linear subspace of H. Often the domain $D(T)$ is a dense subspace of H, in which case T is known as a densely defined operator.

The adjoint of a densely defined unbounded operator is defined in essentially the same manner as for bounded operators. Self-adjoint unbounded operators play the role of the *observables* in the mathematical formulation of quantum mechanics. Examples of self-adjoint unbounded operators on the Hilbert space $L^2(\mathbf{R})$ are:[53]

- A suitable extension of the differential operator

  $$(Af)(x) = i\frac{d}{dx}f(x),$$

  where i is the imaginary unit and f is a differentiable function of compact support.

- The multiplication-by-x operator:

  $$(Bf)(x) = xf(x).$$

These correspond to the momentum and position observables, respectively. Note that neither A nor B is defined on all of H, since in the case of A the derivative need not exist, and in the case of B the product function need not be square integrable. In both cases, the set of possible arguments form dense subspaces of $L^2(\mathbf{R})$.

Constructions

Direct sums

Two Hilbert spaces H_1 and H_2 can be combined into another Hilbert space, called the (orthogonal) direct sum,[54] and denoted

$$H_1 \oplus H_2,$$

consisting of the set of all ordered pairs (x_1, x_2) where $x_i \in H_i$, $i = 1,2$, and inner product defined by

$$\langle (x_1,x_2),(y_1,y_2)\rangle_{H_1\oplus H_2} = \langle x_1,y_1\rangle_{H_1} + \langle x_2,y_2\rangle_{H_2}.$$

More generally, if H_i is a family of Hilbert spaces indexed by $i \in I$, then the direct sum of the H_i, denoted

$$\bigoplus_{i\in I} H_i$$

consists of the set of all indexed families

$$x = (x_i \in H_i | i \in I) \in \prod_{i\in I} H_i$$

in the Cartesian product of the H_i such that

$$\sum_{i\in I} \|x_i\|^2 < \infty.$$

The inner product is defined by

$$\langle x,y\rangle = \sum_{i\in I} \langle x_i, y_i\rangle_{H_i}.$$

Each of the H_i is included as a closed subspace in the direct sum of all of the H_i. Moreover, the H_i are pairwise orthogonal. Conversely, if there is a system of closed subspaces V_i, $i \in I$, in a Hilbert space H which are pairwise orthogonal and whose union is dense in H, then H is canonically isomorphic to the direct sum of V_i. In this case, H is called the internal direct sum of the V_i. A direct sum (internal or external) is also equipped with a family of orthogonal projections E_i onto the ith direct summand H_i. These projections are bounded, self-adjoint, idempotent operators which satisfy the orthogonality condition

$$E_i E_j = 0, \quad i \neq j.$$

The spectral theorem for compact self-adjoint operators on a Hilbert space *H* states that *H* splits into an orthogonal direct sum of the eigenspaces of an operator, and also gives an explicit decomposition of the operator as a sum of projections onto the eigenspaces. The direct sum of Hilbert spaces also appears in quantum mechanics as the Fock space of a system containing a variable number of particles, where each Hilbert space in the direct sum corresponds to an additional degree of freedom for the quantum mechanical system. In representation theory, the Peter-Weyl theorem guarantees that any unitary representation of a compact group on a Hilbert space splits as the direct sum of finite-dimensional representations.

Tensor products

If H_1 and H_2, then one defines an inner product on the (ordinary) tensor product as follows. On simple tensors, let

$$\langle x_1 \otimes x_2,\, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \, \langle x_2, y_2 \rangle.$$

This formula then extends by sesquilinearity to an inner product on $H_1 \otimes H_2$. The Hilbertian tensor product of H_1 and H_2, sometimes denoted by $H_1 \widehat{\otimes} H_2$, is the Hilbert space obtained by completing $H_1 \otimes H_2$for the metric associated to this inner product.[55]

An example is provided by the Hilbert space $L^2([0, 1])$. The Hilbertian tensor product of two copies of $L^2([0, 1])$ is isometrically and linearly isomorphic to the space $L^2([0, 1]^2)$ of square-integrable functions on the square $[0, 1]^2$. This isomorphism sends a simple tensor $f_1 \otimes f_2$to the function

$$(s, t) \to f_1(s)\, f_2(t)$$

on the square.

This example is typical in the following sense.[56] Associated to every simple tensor product $x_1 \otimes x_2$is the rank one operator

$$x^* \in H_1^* \to x^*(x_1)\, x_2$$

from the (continuous) dual H_1^* to H_2. This mapping defined on simple tensors extends to a linear identification between $H_1 \otimes H_2$and the space of finite rank operators from H_1^* to H_2. This extends to a linear isometry of the Hilbertian tensor product $H_1 \widehat{\otimes} H_2$with the Hilbert space $HS(H_1^*, H_2)$ of Hilbert-Schmidt operators from H_1^* to H_2.

Orthonormal bases

The notion of an orthonormal basis from linear algebra generalizes over to the case of Hilbert spaces.[57] In a Hilbert space *H*, an orthonormal basis is a family $\{e_k\}_{k \in B}$ of elements of *H* satisfying the conditions:

1. *Orthogonality*: Every two different elements of *B* are orthogonal: $\langle e_k, e_j \rangle = 0$ for all *k*, *j* in *B* with $k \neq j$.
2. *Normalization*: Every element of the family has norm 1:$\|e_k\| = 1$ for all *k* in *B*.
3. *Completeness*: The linear span of the family e_k, $k \in B$, is dense in *H*.

A system of vectors satisfying the first two conditions basis is called an orthonormal system or an orthonormal set (or an orthonormal sequence if *B* is countable). Such a system is always linearly independent. Completeness of an orthonormal system of vectors of a Hilbert space can be equivalently restated as:

if $\langle v, e_k \rangle = 0$ for all $k \in B$ and some $v \in H$ then $v = \mathbf{0}$.

This is related to the fact that the only vector orthogonal to a dense linear subspace is the zero vector, for if *S* is any orthonormal set and *v* is orthogonal to *S*, then *v* is orthogonal to the closure of the linear span of *S*, which is the whole space.

Examples of orthonormal bases include:

- the set {(1,0,0), (0,1,0), (0,0,1)} forms an orthonormal basis of $\mathbf{R}^3$ with the dot product;

- the sequence $\{f_n : n \in \mathbf{Z}\}$ with $f_n(x) = \exp(2\pi inx)$ forms an orthonormal basis of the complex space $L^2([0,1])$;

In the infinite-dimensional case, an orthonormal basis will not be a basis in the sense of linear algebra; to distinguish the two, the latter basis is also called a Hamel basis. That the span of the basis vectors is dense implies that every vector in the space can be written as the sum of an infinite series, and the orthogonality implies that this decomposition is unique.

Sequence spaces

The space ℓ^2 of square-summable sequences of complex numbers has an orthonormal basis

$$e_1 = (1, 0, 0, \dots)$$
$$e_2 = (0, 1, 0, \dots)$$
$$\vdots$$

More generally, if B is any set, then one can form a Hilbert space of sequences with index set B, defined by

$$\ell^2(B) = \left\{x : B \xrightarrow{x} \mathbb{C} \mid \sum_{b\in B} |x(b)|^2 < \infty\right\}.$$

The summation over B is here defined by

$$\sum_{b\in B} |x(b)|^2 = \sup \sum_{n=1}^{N} |x(b_n)|^2$$

the supremum being taken over all finite subsets of B. It follows that, in order for this sum to be finite, every element of $\ell^2(B)$ has only countably many nonzero terms. This space becomes a Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{b\in B} x(b)\overline{y(b)}$$

for all x and y in $\ell^2(B)$. Here the sum also has only countably many nonzero terms, and is unconditionally convergent by the Cauchy–Schwarz inequality.

An orthonormal basis of $\ell^2(B)$ is indexed by the set B, given by

$$e_b(b') = \begin{cases} 1 & \text{if } \mathrm{b} = \mathrm{b}' \\ 0 & \text{otherwise.} \end{cases}$$

Bessel's inequality and Parseval's formula

Let $f_1, \dots, f_n$ be a finite orthonormal system in H. For an arbitrary vector x in H, let

$$y = \sum_{j=1}^{n} \langle x, f_j\rangle\, f_j.$$

Then $\langle x, f_k\rangle = \langle y, f_k\rangle$ for every $k = 1, \dots, n$. It follows that $x - y$ is orthogonal to each f_k, hence $x - y$ is orthogonal to y. Using the Pythagorean identity twice, it follows that

$$\|x\|^2 = \|x - y\|^2 + \|y\|^2 \geq \|y\|^2 = \sum_{j=1}^{n} |\langle x, f_j\rangle|^2.$$

Let $\{f_i\}$, $i \in I$, be an arbitrary orthonormal system in H. Applying the preceding inequality to every finite subset J of I gives the *Bessel inequality*[58]

$$\sum_{i\in I} |\langle x, f_i\rangle|^2 \leq \|x\|^2, \quad x \in H$$

(according to the definition of the sum of an arbitrary family of non-negative real numbers).

Geometrically, Bessel's inequality implies that the orthogonal projection of x onto the linear subspace spanned by the f_i has norm that does not exceed that of x. In two dimensions, this is the assertion that the length of the leg of a right

triangle may not exceed the length of the hypotenuse.

Bessel's inequality is a stepping stone to the more powerful Parseval identity which governs the case when Bessel's inequality is actually an equality. If $\{e_k\}_{k \in B}$ is an orthonormal basis of *H*, then every element *x* of *H* may be written as

$$x = \sum_{k \in B} \langle x, e_k \rangle \, e_k.$$

Even if *B* is uncountable, Bessel's inequality guarantees that the expression is well-defined and consists only of countably many nonzero terms. This sum is called the *Fourier expansion* of *x*, and the individual coefficients $\langle x, e_k \rangle$ are the *Fourier coefficients* of *x*. Parseval's formula is then

$$\|x\|^2 = \sum_{k \in B} |\langle x, e_k \rangle|^2.$$

Conversely, if $\{e_k\}$ is an orthonormal set such that Parseval's identity holds for every *x*, then $\{e_k\}$ is an orthonormal basis.

Hilbert dimension

As a consequence of Zorn's lemma, *every* Hilbert space admits an orthonormal basis; furthermore, any two orthonormal bases of the same space have the same cardinality, called the Hilbert dimension of the space.[59] For instance, since $\ell^2(B)$ has an orthonormal basis indexed by *B*, its Hilbert dimension is the cardinality of *B* (which may be a finite integer, or a countable or uncountable cardinal number).

As a consequence of Parseval's identity, if $\{e_k\}_{k \in B}$ is an orthonormal basis of *H*, then the map $\Phi : H \to \ell^2(B)$ defined by $\Phi(x) = (\langle x, e_k \rangle)_{k \in B}$ is an isometric isomorphism of Hilbert spaces: it is a bijective linear mapping such that

$$\langle \Phi(x), \Phi(y) \rangle_{\ell^2(B)} = \langle x, y \rangle_H$$

for all *x* and *y* in *H*. The cardinal number of *B* is the Hilbert dimension of *H*. Thus every Hilbert space is isometrically isomorphic to a sequence space $\ell^2(B)$ for some set *B*.

Separable spaces

A Hilbert space is separable if and only if it admits a countable orthonormal basis. All infinite-dimensional separable Hilbert spaces are therefore isometrically isomorphic to ℓ^2.

In the past, Hilbert spaces were often required to be separable as part of the definition.[60] Most spaces used in physics are separable, and since these are all isomorphic to each other, one often refers to any infinite-dimensional separable Hilbert space as "*the* Hilbert space" or just "Hilbert space".[61] Even in quantum field theory, most of the Hilbert spaces are in fact separable, as stipulated by the Wightman axioms. However, it is sometimes argued that non-separable Hilbert spaces are also important in quantum field theory, roughly because the systems in the theory possess an infinite number of degrees of freedom and any infinite Hilbert tensor product (of spaces of dimension greater than one) is non-separable.[62] For instance, a bosonic field can be naturally thought of as an element of a tensor product whose factors represent harmonic oscillators at each point of space. From this perspective, the natural state space of a boson might seem to be a non-separable space.[62] However, it is only a small separable subspace of the full tensor product that can contain physically meaningful fields (on which the observables can be defined). Another non-separable Hilbert space models the state of an infinite collection of particles in an unbounded region of space. An orthonormal basis of the space is indexed by the density of the particles, a continuous parameter, and since the set of possible densities is uncountable, the basis is not countable.[62]

Orthogonal complements and projections

If S is a subset of a Hilbert space H, the set of vectors orthogonal to S is defined by

$$S^\perp = \{x \in H : \langle x, s\rangle = 0\ \forall s \in S\}.$$

$S^\perp$ is a closed subspace of H and so forms itself a Hilbert space. If V is a closed subspace of H, then $V^\perp$ is called the *orthogonal complement* of V. In fact, every x in H can then be written uniquely as $x = v + w$, with v in V and w in $V^\perp$. Therefore, H is the internal Hilbert direct sum of V and $V^\perp$.

The linear operator $P_V : H \to H$ which maps x to v is called the *orthogonal projection* onto V. There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all bounded self-adjoint operators P such that $P^2 = P$. Specifically,

> **Theorem**. The orthogonal projection P_V is a self-adjoint linear operator on H of norm ≤ 1 with the property $P^2_V = P_V$. Moreover, any self-adjoint linear operator E such that $E^2 = E$ is of the form P_V, where V is the range of E. For every x in H, $P_V(x)$ is the unique element v of V which minimizes the distance $\|x - v\|$.

This provides the geometrical interpretation of $P_V(x)$: it is the best approximation to x by elements of V.[63]

An operator P such that $P = P^2 = P^*$ is called an orthogonal projection. The orthogonal projection P_V onto a closed subspace V of H is the adjoint of the inclusion mapping

$$i_V : V \to H,$$

meaning that

$$\langle i_V x, y\rangle = \langle x, P_V y\rangle$$

for all $x \in H$ and $y \in V$. Projections P_U and P_V are called mutually orthogonal if $P_U P_V = 0$. This is equivalent to U and V being orthogonal as subspaces of H. As a result, the sum of the two projections P_U and P_V is only a projection if U and V are orthogonal to each other, and in that case $P_U + P_V = P_{U+V}$. The composite $P_U P_V$ is generally not a projection; in fact, the composite is a projection if and only if the two projections commute, and in that case $P_U P_V = P_{U\cap V}$.

The operator norm of a projection P onto a non-zero closed subspace is equal to one:

$$\|P\| = \sup_{x\in H, x\neq 0} \frac{\|Px\|}{\|x\|} = 1.$$

Every closed subspace V of a Hilbert space is therefore the image of an operator P of norm one such that $P^2 = P$. In fact this property characterizes Hilbert spaces:[64]

- A Banach space of dimension higher than 2 is (isometrically) a Hilbert space if and only if, to every closed subspace V, there is an operator P_V of norm one whose image is V such that $P_V^2 = P_V$.

While this result characterizes the metric structure of a Hilbert space, the structure of a Hilbert space as a topological vector space can itself be characterized in terms of the presence of complementary subspaces:[65]

- A Banach space X is topologically and linearly isomorphic to a Hilbert space if and only if, to every closed subspace V, there is a closed subspace W such that X is equal to the internal direct sum $V \oplus W$.

The orthogonal complement satisfies some more elementary results. It is a monotone function in the sense that if $U \subset V$, then $V^\perp \subseteq U^\perp$ with equality holding if and only if V is contained in the closure of U. This result is a special case of the Hahn-Banach theorem. The closure of a subspace can be completely characterized in terms of the orthogonal complement: If V is a subspace of H, then the closure of V is equal to $V^{\perp\perp}$. The orthogonal complement is thus a Galois connection on the partial order of subspaces of a Hilbert space. In general, the orthogonal complement of a sum of subspaces is the intersection of the orthogonal complements:[66] $\left(\sum_i V_i\right)^\perp = \bigcap_i V_i^\perp$. If the V_i are in addition closed, then $\overline{\sum_i V_i^\perp} = \left(\bigcap_i V_i\right)^\perp$.

Spectral theory

There is a well-developed spectral theory for self-adjoint operators in a Hilbert space, that is roughly analogous to the study of symmetric matrices over the reals or self-adjoint matrices over the complex numbers.[67] In the same sense, one can obtain a "diagonalization" of a self-adjoint operator as a suitable sum (actually an integral) of orthogonal projection operators.

The spectrum of an operator *T*, denoted σ(*T*) is the set of complex numbers λ such that *T* – λ lacks a continuous inverse. If *T* is bounded, then the spectrum is always a compact set in the complex plane, and lies inside the disc $|z|\leq\|T\|$. If *T* is self-adjoint, then the spectrum is real. In fact, it is contained in the interval [*m*,*M*] where

$$m = \inf_{\|x\|=1}\langle Tx, x\rangle,\quad M = \sup_{\|x\|=1}\langle Tx, x\rangle.$$

Moreover, *m* and *M* are both actually contained within the spectrum.

The eigenspaces of an operator *T* are given by

$$H_\lambda = \ker(T-\lambda).$$

Unlike with finite matrices, not every element of the spectrum of *T* must be an eigenvalue: the linear operator *T* – λ may only lack an inverse because it is not surjective. Elements of the spectrum of an operator in the general sense are known as *spectral values*. Since spectral values need not be eigenvalues, the spectral decomposition is often more subtle than in finite dimensions.

However, the spectral theorem of a self-adjoint operator *T* takes a particularly simple form if, in addition, *T* is assumed to be a compact operator. The spectral theorem for compact self-adjoint operators states:[68]

- A compact self-adjoint operator *T* has only countably (or finitely) many spectral values. The spectrum of *T* has no limit point in the complex plane except possibly zero. The eigenspaces of *T* decompose *H* into an orthogonal direct sum:

  $$H = \bigoplus_{\lambda\in\sigma(T)} H_\lambda.$$

  Moreover, if E_λ denotes the orthogonal projection onto the eigenspace H_λ, then

  $$T = \sum_{\lambda\in\sigma(T)} \lambda E_\lambda$$

  where the sum converges with respect to the norm on B(*H*).

This theorem plays a fundamental role in the theory of integral equations, as many integral operators are compact, in particular those that arise from Hilbert-Schmidt operators.

The general spectral theorem for self-adjoint operators involves a kind of operator-valued Riemann-Stieltjes integral, rather than an infinite summation.[69] The *spectral family* associated to *T* associates to each real number λ an operator E_λ, which is the projection onto the nullspace of the operator $(T-\lambda)^+$, where the positive part of a self-adjoint operator is defined by

$$A^+ = \frac{1}{2}\left(\sqrt{A^2}+A\right).$$

The operators E_λ are monotone increasing relative to the partial order defined on self-adjoint operators; the eigenvalues correspond precisely to the jump discontinuities. One has the spectral theorem, which asserts

$$T = \int_{\mathbb{R}} \lambda\, dE_\lambda.$$

The integral is understood as a Riemann-Stieltjes integral, convergent with respect to the norm on B(*H*). In particular, one has the ordinary scalar-valued integral representation

$$\langle Tx, y\rangle = \int_{\mathbb{R}} \lambda\, d\langle E_\lambda x, y\rangle.$$

A somewhat similar spectral decomposition holds for normal operators, although because the spectrum may now contain non-real complex numbers, the operator-valued Stieltjes measure dE_λ must instead be replaced by a resolution of the identity.

A major application of spectral methods is the spectral mapping theorem, which allows one to apply to a self-adjoint operator T any continuous complex function f defined on the spectrum of T by forming the integral

$$f(T) = \int_{\sigma(T)} f(\lambda)\, dE_\lambda.$$

The resulting continuous functional calculus has applications in particular to pseudodifferential operators.[70]

The spectral theory of *unbounded* self-adjoint operators is only marginally more difficult than for bounded operators. The spectrum of an unbounded operator is defined in precisely the same way as for bounded operators: λ is a spectral value if the resolvent operator

$$R_\lambda = (T - \lambda)^{-1}$$

fails to be a well-defined continuous operator. The self-adjointness of T still guarantees that the spectrum is real. Thus the essential idea of working with unbounded operators is to look instead at the resolvent R_λ where λ is non-real. This is a *bounded* normal operator, which admits a spectral representation that can then be transferred to a spectral representation of T itself. A similar strategy is used, for instance, to study the spectrum of the Laplace operator: rather than address the operator directly, one instead looks as an associated resolvent such as a Riesz potential or Bessel potential.

A precise version of the spectral theorem which holds in this case is:[71]

> Given a densely-defined self-adjoint operator T on a Hilbert space H, there corresponds a unique resolution of the identity E on the Borel sets of **R**, such that
>
> $$\langle Tx, y\rangle = \int_{\mathbb{R}} \lambda\, dE_{x,y}(\lambda)$$
>
> for all $x \in D(T)$ and $y \in H$. The spectral measure E is concentrated on the spectrum of T.

There is also a version of the spectral theorem that applies to unbounded normal operators.

See also

- Harmonic analysis
- Hermitian operators
- Hilbert C*-module
- Hilbert algebra
- Hilbert manifold
- Rigged Hilbert space
- Topologies on the set of operators on a Hilbert space

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External links

- Hilbert Space at Mathworld [93]
- 245B, notes 5: Hilbert spaces [94] by Terence Tao

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[2] The mathematical material in this article can be found in any good textbook on functional analysis, such as Dieudonné (1960), Hewitt & Stromberg (1965), Reed & Simon (1980) or Rudin (1980).

[3] Dieudonné 1960, §6.2

[4] Dieudonné 1960

[5] Largely from the work of Hermann Grassmann, at the urging of August Ferdinand Möbius (Boyer & Merzbach 1991, pp. 584-586). The first modern axiomatic account of abstract vector spaces ultimately appeared in Giuseppe Peano's 1888 account (Grattan-Guinness 2000, §5.2.2; O'Connor & Robertson 1996).

[6] A detailed account of the history of Hilbert spaces can be found in Bourbaki 1987.

[7] Schmidt 1908

[8] Titchmarsh 1946, §IX.1

[9] Lebesgue 1904. Further details on the history of integration theory can be found in Bourbaki (1987) and Saks (2005).

[10] Bourbaki 1987.

[11] Dunford & Schwartz 1958, §IV.16

[12] In Dunford & Schwartz (1958, §IV.16), the result that every linear functional on $L^2[0,1]$ is represented by integration is jointly attributed to Fréchet (1907) and Riesz (1907). The general result, that the dual of a Hilbert space is identified with the Hilbert space itself, can be found in Riesz (1934).

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[14] Kline 1972, p. 1092

[15] Hilbert, Nordheim & von Neumann 1927.

[16] Weyl 1931.

[17] Prugovečki 1981, pp. 1–10.

[18] von Neumann 1932

[19] Halmos 1957, Section 42.

[20] Hewitt & Stromberg 1965.

[21] Bers, John & Schechter 1981.

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[23] Stein 1970

[24] Details can be found in Warner (1983).

[25] A general reference on Hardy spaces is the book Duren (1970).

[26] Krantz 2002, §1.4

[27] Krantz 2002, §1.5

[28] Young 1987, Chapter 9.

[29] The eigenvalues of the Fredholm kernel are $1/\lambda$, which tend to zero.

[30] More detail on finite element methods from this point of view can be found in Brenner & Scott (2005).

[31] von Neumann 1932

[32] Reed & Simon 1980

[33] A treatment of Fourier series from this point of view is available, for instance, in Rudin (1987).

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[38] Kac 1966

[39] Dirac 1930

[40] von Neumann 1955

[41] Young 1988, p. 23.

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[45] Rudin 1987, Theorem 4.11
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[47] Weidmann 1980, §4.5
[48] Buttazzo, Giaquinta & Hildebrandt 1998, Theorem 5.17
[49] Halmos 1982, Problem 52, 58
[50] Rudin 1973
[51] Trèves 1967, Chapter 18
[52] See Prugovečki (1981), Reed & Simon (1980, Chapter VIII) and Folland (1989).
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[57] Dunford & Schwartz 1958, §IV.4.
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[64] Kakutani 1939
[65] Lindenstrauss & Tzafriri 1971
[66] Halmos 1957, §12
[67] A general account of spectral theory in Hilbert spaces can be found in Riesz & Sz Nagy (1990). A more sophisticated account in the language of C^*-algebras is in Rudin (1973) or Kadison & Ringrose (1997)
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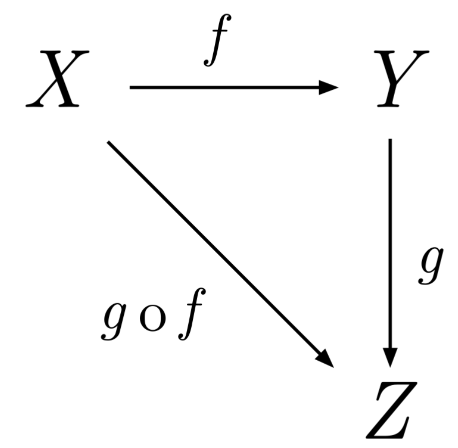
Categorical and Topological Dynamics. Category Theory and Categorical Dynamics Concepts

Category theory

In mathematics, **category theory** deals in an abstract way with mathematical structures and relationships between them: it abstracts from *sets* and *functions* to *objects* linked in diagrams by *morphisms* or *arrows*.

One of the simplest examples of a category (which is a very important concept in topology) is that of groupoid, defined as a category whose arrows or morphisms are all invertible. Categories now appear in most branches of mathematics, some areas of theoretical computer science where they correspond to types, and mathematical physics where they can be used to describe vector spaces. Category theory provides both with a unifying notion and terminology. Categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in 1942–45, in connection with → algebraic topology.



A category with objects X, Y, Z and morphisms *f*, *g*

Category theory has several faces known not just to specialists, but to other mathematicians. A term dating from the 1940s, "general abstract nonsense", refers to its high level of abstraction, compared to more classical branches of mathematics. Homological algebra is category theory in its aspect of organising and suggesting manipulations in abstract algebra. Diagram chasing is a visual method of arguing with abstract "arrows" joined in diagrams. Note that arrows between categories are called functors, subject to specific defining commutativity conditions; moreover, categorical diagrams and sequences can be defined as functors (viz. Mitchell, 1965). An arrow between two functors is a natural transformation when it is subject to certain naturality or commutativity conditions. Both functors and natural transformations are key concepts in category theory, or the "real engines" of category theory. To paraphrase a famous sentence of the mathematicians who founded category theory: 'Categories were introduced to define functors, and functors were introduced to define natural transformations'. Topos theory is a form of abstract sheaf theory, with geometric origins, and leads to ideas such as pointless topology. A topos can also be considered as a specific type of category with two additional topos axioms.

Background

The study of categories is an attempt to *axiomatically* capture what is commonly found in various classes of related *mathematical structures* by relating them to the *structure-preserving functions* between them. A systematic study of category theory then allows us to prove general results about any of these types of mathematical structures from the axioms of a category.

Consider the following example. The class **Grp** of groups consists of all objects having a "group structure". One can proceed to prove theorems about groups by making logical deductions from the set of axioms. For example, it is immediately proved from the axioms that the identity element of a group is unique.

Instead of focusing merely on the individual objects (e.g., groups) possessing a given structure, category theory emphasizes the morphisms – the structure-preserving mappings – *between* these objects; by studying these morphisms, we are able to learn more about the structure of the objects. In the case of groups, the morphisms are the group homomorphisms. A group homomorphism between two groups "preserves the group structure" in a precise sense – it is a "process" taking one group to another, in a way that carries along information about the structure of the first group into the second group. The study of group homomorphisms then provides a tool for studying general properties of groups and consequences of the group axioms.

A similar type of investigation occurs in many mathematical theories, such as the study of continuous maps (morphisms) between topological spaces in topology (the associated category is called **Top**), and the study of smooth functions (morphisms) in manifold theory.

If one axiomatizes relations instead of functions, one obtains the theory of allegories.

Functors

Abstracting again, a category is *itself* a type of mathematical structure, so we can look for "processes" which preserve this structure in some sense; such a process is called a functor. A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second.

In fact, what we have done is define a category *of categories and functors* – the objects are categories, and the morphisms (between categories) are functors.

By studying categories and functors, we are not just studying a class of mathematical structures and the morphisms between them; we are studying the *relationships between various classes of mathematical structures*. This is a fundamental idea, which first surfaced in → algebraic topology. Difficult *topological* questions can be translated into *algebraic* questions which are often easier to solve. Basic constructions, such as the fundamental group or fundamental groupoid [1] of a topological space, can be expressed as fundamental functors [1] to the category of groupoids in this way, and the concept is pervasive in algebra and its applications.

Natural transformation

Abstracting yet again, constructions are often "naturally related" – a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to "map" one functor to another. Many important constructions in mathematics can be studied in this context. "Naturality" is a principle, like general covariance in physics, that cuts deeper than is initially apparent.

Historical notes

In 1942–45, Samuel Eilenberg and Saunders Mac Lane were the first to introduce categories, functors, and natural transformations as part of their work in topology, especially → algebraic topology. Their work was an important part of the transition from intuitive and geometric homology to axiomatic homology theory. Eilenberg and Mac Lane later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories.

Stanislaw Ulam, and some writing on his behalf, have claimed that related ideas were current in the late 1930s in Poland. Eilenberg was Polish, and studied mathematics in Poland in the 1930s. Category theory is also, in some sense, a continuation of the work of Emmy Noether (one of Mac Lane's teachers) in formalizing abstract processes; Noether realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Eilenberg and Mac Lane proposed an axiomatic formalization of the relation between structures and the processes preserving them.

The subsequent development of category theory was powered first by the computational needs of homological algebra, and later by the axiomatic needs of algebraic geometry, the field most resistant to being grounded in either axiomatic set theory or the Russell-Whitehead view of united foundations. General category theory, an extension of universal algebra having many new features allowing for semantic flexibility and higher-order logic, came later; it is now applied throughout mathematics.

Certain categories called topoi (singular *topos*) can even serve as an alternative to axiomatic set theory as a foundation of mathematics. These foundational applications of category theory have been worked out in fair detail as a basis for, and justification of, constructive mathematics. More recent efforts to introduce undergraduates to categories as a foundation for mathematics include Lawvere and Rosebrugh (2003) and Lawvere and Schanuel (1997).

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with applications in functional programming and domain theory, where a cartesian closed category is taken as a non-syntactic description of a lambda calculus. At the very least, category theoretic language clarifies what exactly these related areas have in common (in some abstract sense).

Categories, objects and morphisms

A *category* C consists of the following three mathematical entities:

- A class ob(C), whose elements are called *objects*;
- A class hom(C), whose elements are called morphisms or maps or *arrows*. Each morphism f has a unique *source object* a and *target object* b. We write f: $a \to b$, and we say "f is a morphism from a to b". We write hom(a, b) (or Hom(a, b), or $\hom_C(a, b)$, or Mor(a, b), or $C(a, b)$) to denote the *hom-class* of all morphisms from a to b.
- A binary operation $\circ$, called *composition of morphisms*, such that for any three objects a, b, and c, we have hom(a, b) × hom(b, c) → hom(a, c). The composition of f: $a \to b$ and g: $b \to c$ is written as $g \circ f$ or gf[2], governed by two axioms:
  - Associativity: If $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
  - Identity: For every object x, there exists a morphism $1_x : x \to x$ called the *identity morphism for x*, such that for every morphism $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.

From these axioms, it can be proved that there is exactly one identity morphism for every object. Some authors deviate from the definition just given by identifying each object with its identity morphism.

Relations among morphisms (such as $fg = h$) are often depicted using commutative diagrams, with "points" (corners) representing objects and "arrows" representing morphisms.

Properties of morphisms

Some morphisms have important properties. A morphism $f : a \to b$ is:

- a monomorphism (or *monic*) if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : x \to a$.
- an epimorphism (or *epic*) if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : b \to x$.
- an isomorphism if there exists a morphism $g : b \to a$ with $f \circ g = 1_b$ and $g \circ f = 1_a$.[3]
- an endomorphism if $a = b$. end(a) denotes the class of endomorphisms of a.
- an automorphism if f is both an endomorphism and an isomorphism. aut(a) denotes the class of automorphisms of a.

Functors

Functors are structure-preserving maps between categories. They can be thought of as morphisms in the category of all (small) categories.

A (**covariant**) functor F from a category C to a category D, written $F{:}C \to D$, consists of:

- for each object x in C, an object $F(x)$ in D; and
- for each morphism $f : x \to y$ in C, a morphism $F(f) : F(x) \to F(y)$,

such that the following two properties hold:

- For every object x in C, $F(1_x) = 1_{F(x)}$;
- For all morphisms $f : x \to y$ and $g : y \to z$, $F(g \circ f) = F(g) \circ F(f)$.

A **contravariant** functor F: $C \to D$, is like a covariant functor, except that it "turns morphisms around" ("reverses all the arrows"). More specifically, every morphism $f : x \to y$ in C must be assigned to a morphism $F(f) : F(y) \to F(x)$ in D. In other words, a contravariant functor is a covariant functor from the opposite category C^{op} to D.

Natural transformations and isomorphisms

A *natural transformation* is a relation between two functors. Functors often describe "natural constructions" and natural transformations then describe "natural homomorphisms" between two such constructions. Sometimes two quite different constructions yield "the same" result; this is expressed by a natural isomorphism between the two functors.

If F and G are (covariant) functors between the categories C and D, then a natural transformation from F to G associates to every object x in C a morphism $\eta_x : F(x) \to G(x)$ in D such that for every morphism $f : x \to y$ in C, we have $\eta_y \circ F(f) = G(f) \circ \eta_x$; this means that the following diagram is commutative:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

The two functors F and G are called *naturally isomorphic* if there exists a natural transformation from F to G such that η_x is an isomorphism for every object x in C.

Universal constructions, limits, and colimits

Using the language of category theory, many areas of mathematical study can be cast into appropriate categories, such as the categories of all sets, groups, topologies, and so on. These categories surely have some objects that are "special" in a certain way, such as the empty set or the product of two topologies, yet in the definition of a category, objects are considered to be atomic, i.e., we *do not know* whether an object A is a set, a topology, or any other abstract concept – hence, the challenge is to define special objects without referring to the internal structure of those objects. But how can we define the empty set without referring to elements, or the product topology without referring to open sets?

The solution is to characterize these objects in terms of their relations to other objects, as given by the morphisms of the respective categories. Thus, the task is to find *universal properties* that uniquely determine the objects of interest. Indeed, it turns out that numerous important constructions can be described in a purely categorical way. The central concept which is needed for this purpose is called categorical *limit*, and can be dualized to yield the notion of a *colimit*.

Equivalent categories

It is a natural question to ask: under which conditions can two categories be considered to be "essentially the same", in the sense that theorems about one category can readily be transformed into theorems about the other category? The major tool one employs to describe such a situation is called *equivalence of categories*, which is given by appropriate functors between two categories. Categorical equivalence has found numerous applications in mathematics.

Further concepts and results

The definitions of categories and functors provide only the very basics of categorical algebra; additional important topics are listed below. Although there are strong interrelations between all of these topics, the given order can be considered as a guideline for further reading.

- The functor category D^C has as objects the functors from C to D and as morphisms the natural transformations of such functors. The Yoneda lemma is one of the most famous basic results of category theory; it describes representable functors in functor categories.
- Duality: Every statement, theorem, or definition in category theory has a *dual* which is essentially obtained by "reversing all the arrows". If one statement is true in a category C then its dual will be true in the dual category C^{op}. This duality, which is transparent at the level of category theory, is often obscured in applications and can lead to surprising relationships.
- Adjoint functors: A functor can be left (or right) adjoint to another functor that maps in the opposite direction. Such a pair of adjoint functors typically arises from a construction defined by a universal property; this can be seen as a more abstract and powerful view on universal properties.

Higher-dimensional categories

Many of the above concepts, especially equivalence of categories, adjoint functor pairs, and functor categories, can be situated into the context of *higher-dimensional categories*. Briefly, if we consider a morphism between two objects as a "process taking us from one object to another", then higher-dimensional categories allow us to profitably generalize this by considering "higher-dimensional processes".

For example, a (strict) 2-category is a category together with "morphisms between morphisms", i.e., processes which allow us to transform one morphism into another. We can then "compose" these "bimorphisms" both horizontally and vertically, and we require a 2-dimensional "exchange law" to hold, relating the two composition laws. In this context, the standard example is **Cat**, the 2-category of all (small) categories, and in this example, bimorphisms of morphisms are simply natural transformations of morphisms in the usual sense. Another basic example is to consider a 2-category with a single object; these are essentially monoidal categories. Bicategories are a weaker notion of 2-dimensional categories in which the composition of morphisms is not strictly associative, but only associative "up to" an isomorphism.

This process can be extended for all natural numbers n, and these are called n-categories. There is even a notion of *ω-category* corresponding to the ordinal number ω.

Higher-dimensional categories are part of the broader mathematical field of higher-dimensional algebra,a concept introduced by → Ronald Brown. For a conversational introduction to these ideas, see John Baez, 'A Tale of n-categories' (1996). [4]

See also

- Important publications in category theory
- Glossary of category theory
- Domain theory
- Enriched category theory
- Higher category theory
- Timeline of category theory and related mathematics
- Higher-dimensional algebra

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- Homepage of the Categories mailing list, [18] with extensive resource list.
- Baez, John, 1996,"The Tale of *n*-categories. [4]" An informal introduction to higher order categories.
- The catsters [19]" a Youtube channel about category theory.
- Category Theory [20] on PlanetMath
- Categories, Logic and the Foundations of Physics [21], Webpage dedicated to the use of Categories and Logic in the Foundations of Physics.
- Interactive Web page [22] which generates examples of categorical constructions in the category of finite sets. Written by Jocelyn Paine [23]

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[1] http://planetphysics.org/encyclopedia/FundamentalGroupoidFunctor.html

[2] Some authors compose in the opposite order, writing *fg* or $f \circ g$ for $g \circ f$. Computer scientists using category theory very commonly write *f;g* for $g \circ f$

[3] Note that a morphism that is both epic and monic is not necessarily an isomorphism! For example, in the category consisting of two objects *A* and *B*, the identity morphisms, and a single morphism *f* from *A* to *B*, *f* is both epic and monic but is not an isomorphism.

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Higher dimensional algebra

This article is about ***higher-dimensional algebra and supercategories*** *in generalized → category theory, super-category theory, and also its extensions in metamathematics*[1] . Supercategories were first introduced in 1970,[2] and were subsequently developed for applications in Theoretical Physics (especially Quantum Field Theory and Topological quantum field theory) and Mathematical Biology or Mathematical Biophysics.[3] In higher-dimensional algebra, a double groupoid is a generalisation of a one-dimensional groupoid to two dimensions[4] , and the latter groupoid can be considered as a special case of a category with all invertible arrows, or morphisms.

Double groupoids are often used to capture information about geometrical objects such as higher-dimensional manifolds (or *n*-dimensional manifolds)[5] . In general, an *n*-dimensional manifold is a space that locally looks like an *n*-dimensional Euclidean space, but whose global structure may be non-Euclidean. A first step towards defining higher dimensional algebras is the concept of 2-category, followed by the more `geometric' concept of double category [6][7] [8] .

A higher level concept is that of a category of categories, or **super-category** which generalises to higher dimensions the notion of category – regarded as any structure which is an interpretation of Lawvere's axioms of the *elementary theory of abstract categories* (ETAC)[9] [10] [11] [12] . Thus, a supercategory and also a super-category, can be regarded as natural extensions of the concepts of meta-category,[13] multicategory, and multi-graph, k-partite graph, or colored graph (see a color figure, and also its definition in graph theory).

Double groupoids were first introduced by → Ronald Brown in 1976, in ref.[14] and were further developed towards applications in nonabelian → algebraic topology[15] [16] [17] [18] .

See also

- Higher category theory
- → Category theory
- → Algebraic topology
- Seifert–van Kampen theorem
- Abstract algebra
- Categorical algebra
- Esquisse d'un Programme
- Grothendieck's Galois theory
- Metatheory
- Metalogic
- Metamathematics
- Colored graphs
- Multicategory
- Enriched category

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Algebraic topology

Algebraic topology is a branch of mathematics which uses tools from abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism. In many situations this is too much to hope for and it is more prudent to aim for a more modest goal, classification up to homotopy equivalence.

Although algebraic topology primarily uses algebra to study topological problems, the converse, using topology to solve algebraic problems, is sometimes also possible. Algebraic topology, for example, allows for a convenient proof that any subgroup of a free group is again a free group.

The method of algebraic invariants

An older name for the subject was combinatorial topology, implying an emphasis on how a space X was constructed from simpler ones (the modern standard tool for such construction is the CW-complex). The basic method now applied in algebraic topology is to investigate spaces via algebraic invariants by mapping them, for example, to groups which have a great deal of manageable structure in a way that respects the relation of homeomorphism (or more general homotopy) of spaces. This allows one to recast statements about topological spaces into statements about groups, which are often easier to prove.

Two major ways in which this can be done are through fundamental groups, or more generally homotopy theory, and through homology and cohomology groups. The fundamental groups give us basic information about the structure of a topological space, but they are often nonabelian and can be difficult to work with. The fundamental group of a (finite) simplicial complex does have a finite presentation.

Homology and cohomology groups, on the other hand, are abelian and in many important cases finitely generated. Finitely generated abelian groups are completely classified and are particularly easy to work with.

Setting in category theory

In general, all constructions of algebraic topology are → functorial; the notions of category, functor and natural transformation originated here. Fundamental groups and homology and cohomology groups are not only *invariants* of the underlying topological space, in the sense that two topological spaces which are homeomorphic have the same associated groups, but their associated morphisms also correspond — a continuous mapping of spaces induces a group homomorphism on the associated groups, and these homomorphisms can be used to show non-existence (or, much more deeply, existence) of mappings.

Results on homology

Several useful results follow immediately from working with finitely generated abelian groups. The free rank of the n-th homology group of a simplicial complex is equal to the n-th Betti number, so one can use the homology groups of a simplicial complex to calculate its Euler-Poincaré characteristic. As another example, the top-dimensional integral homology group of a closed manifold detects orientability: this group is isomorphic to either the integers or 0, according as the manifold is orientable or not. Thus, a great deal of topological information is encoded in the homology of a given topological space.

Beyond simplicial homology, which is defined only for simplicial complexes, one can use the differential structure of smooth manifolds via de Rham cohomology, or Čech or sheaf cohomology to investigate the solvability of differential equations defined on the manifold in question. De Rham showed that all of these approaches were interrelated and that, for a closed, oriented manifold, the Betti numbers derived through simplicial homology were the same Betti numbers as those derived through de Rham cohomology. This was extended in the 1950s, when

Eilenberg and Steenrod generalized this approach. They defined homology and cohomology as functors equipped with natural transformations subject to certain axioms (e.g., a weak equivalence of spaces passes to an isomorphism of homology groups), verified that all existing (co)homology theories satisfied these axioms, and then proved that such an axiomatization uniquely characterized the theory.

A new approach uses a functor from filtered spaces to crossed complexes defined directly and homotopically using relative homotopy groups; a higher homotopy van Kampen theorem proved for this functor enables basic results in algebraic topology, especially on the border between homology and homotopy, to be obtained without using singular homology or simplicial approximation. This approach is also called non abelian algebraic topology, and generalises to higher dimensions ideas coming from the fundamental group.

Applications of algebraic topology

Classic applications of algebraic topology include:

- The Brouwer fixed point theorem: every continuous map from the unit n-disk to itself has a fixed point.
- The n-sphere admits a nowhere-vanishing continuous unit vector field if and only if n is odd. (For n=2, this is sometimes called the "hairy ball theorem".)
- The Borsuk-Ulam theorem: any continuous map from the n-sphere to Euclidean n-space identifies at least one pair of antipodal points.
- Any subgroup of a free group is free. This result is quite interesting, because the statement is purely algebraic yet the simplest proof is topological. Namely, any free group G may be realized as the fundamental group of a graph X. The main theorem on covering spaces tells us that every subgroup H of G is the fundamental group of some covering space Y of X; but every such Y is again a graph. Therefore its fundamental group H is free.
- Topological combinatorics

Notable algebraic topologists

- Frank Adams
- Karol Borsuk
- Luitzen Egbertus Jan Brouwer
- William Browder
- Nicolas Bourbaki
- Henri Cartan
- Otto Hermann Künneth
- Samuel Eilenberg
- Peter Freyd
- Alexander Grothendieck
- Friedrich Hirzebruch
- Heinz Hopf
- Michael J. Hopkins
- Witold Hurewicz
- Egbert van Kampen
- Saunders Mac Lane
- J.P. May
- John Coleman Moore
- Sergei Petrovich Novikov
- Lev Pontryagin
- Daniel Quillen
- Jean-Pierre Serre

- Norman Steenrod
- Dennis Sullivan
- René Thom
- Hassler Whitney
- J. H. C. Whitehead

Important theorems in algebraic topology

- Borsuk-Ulam theorem
- Brouwer fixed point theorem
- Cellular approximation theorem
- Eilenberg–Zilber theorem
- Hurewicz theorem
- Kunneth theorem
- Poincaré duality theorem
- Universal coefficient theorem
- Van Kampen's theorem
- Generalized van Kampen's theorems [1]
- Higher homotopy, generalized van Kampen's theorem[2]
- Whitehead's theorem

See also

- Important publications in algebraic topology
- GNUL Textbook on Algebraic Topology vol.1 [3][4]
- → Higher dimensional algebra
- Higher category theory
- Van Kampen's theorem
- Groupoid
- Lie groupoid
- Lie algebroid
- Grothendieck topology
- Serre spectral sequence
- Sheaf
- Homotopy
- Homotopy theory
- Fundamental group
- Homology theory
- Homological algebra
- Cohomology theory
- K-theory
- Algebraic K-theory
- TQFT
- Homotopy quantum field theory(HQFT)
- CW complex
- Simplicial complex
- Homology complex
- Algebroid

- Exact sequence

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Topological dynamics

In mathematics, **topological dynamics** is a branch of the theory of dynamical systems in which qualitative, asymptotic properties of dynamical systems are studied from the viewpoint of general topology.

Scope

The central object of study in topological dynamics is a **topological dynamical system**, i.e. a topological space, together with a continuous transformation, a continuous flow, or more generally, a semigroup of continuous transformations of that space. The origins of topological dynamics lie in the study of asymptotical properties of trajectories of systems of autonomous ordinary differential equations, in particular, the behavior of limit sets and various manifestations of "repetetiveness" of the motion, such as periodic trajectories, recurrence and minimality, stability, non-wandering points. → George Birkhoff is considered to be the founder of the field. A structure theorem for minimal distal flows proved by Hillel Furstenberg in the early 1960s inspired much work on classification of minimal flows. A lot of research in the 1970s and 1980s was devoted to topological dynamics of one-dimensional maps, in particular, piecewise linear self-maps of the interval and the circle.

Unlike the theory of smooth dynamical systems, where the main object of study is a smooth manifold with a diffeomorphism or a smooth flow, phase spaces considered in topological dynamics are general metric spaces (usually, compact). This necessitates development of entirely different techniques but allows extra degree of flexibility even in the smooth setting, because invariant subsets of a manifold are frequently very complicated topologically (cf limit cycle, strange attractor); additionally, → shift spaces arising via symbolic representations can be considered on an equal footing with more geometric actions. Topological dynamics has intimate connections with → ergodic theory of dynamical systems, and many fundamental concepts of the latter have topological analogues (cf Kolmogorov–Sinai entropy and topological entropy).

See also

- Poincaré–Bendixson theorem
- → Symbolic dynamics
- Topological conjugacy

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Graph dynamical system

In mathematics, the concept of **graph dynamical systems** can be used to capture a wide range of processes taking place on graphs or networks. A major theme in the mathematical and computational analysis of GDSs is to relate their structural properties (e.g. the network connectivity) and the global dynamics that result.

The work on GDSs considers finite graphs and finite state spaces. As such, the research typically involves techniques from, e.g., graph theory, combinatorics, algebra, and dynamical systems rather than differential geometry. In principle, one could define and study GDSs over an infinite graph (e.g. cellular automata over $\mathbb{Z}^k$ or interacting particle systems), as well as GDSs with infinite state space (e.g. $\mathbb{R}$ as in coupled map lattices); see, e.g., Wu[1] . In the following everything is implicitly assumed to be finite unless stated otherwise.

Formal definition

A graph dynamical system is constructed from the following components:

- A finite *graph* Y with vertex set v[Y] = {1,2, ... , n}. Depending on the context the graph can be directed or undirected.
- A state x_v for each vertex v of Y taken from a finite set K. The *system state* is the n-tuple $x = (x_1, x_2, \dots , x_n)$, and $x[v]$ is the tuple consisting of the states associated to the vertices in the 1-neighborhood of v in Y (in some fixed order).
- A *vertex function* f_v for each vertex v. The vertex function maps the state of vertex v at time t to the vertex state at time $t + 1$ based on the states associated to the 1-neighborhood of v in Y.
- An *update scheme* specifying the mechanism by which the mapping of individual vertex states is carried out so as to induce a discrete dynamical system with map $F: K^n \to K^n$.

The *phase space* associated to a dynamical system with map $F: K^n \to K^n$ is the finite directed graph with vertex set K^n and directed edges $(x, F(x))$. The structure of the phase space is governed by the properties of the graph Y, the

vertex functions $(f_i)_i$, and the update scheme. The research in this area seeks to infer phase space properties based on the structure of the system constituents. The analysis has a local-to-global character.

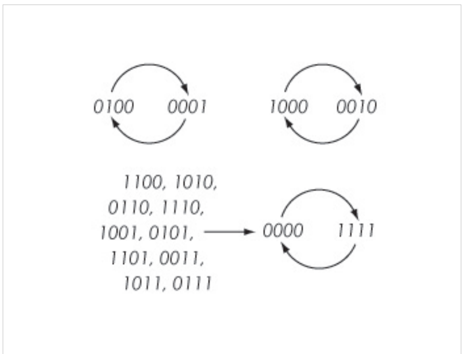
Generalized cellular automata (GCA)

If, for example, the update scheme consists of applying the vertex functions synchronously one obtains the class of *generalized cellular automata* (CA). In this case, the global map F: $K^n \to K^n$ is given by

$$F(x)_v = f_v(x[v]) \ .$$

This class is referred to as generalized cellular automata since the classical or standard cellular automata are typically defined and studied over regular graphs or grids, and the vertex functions are typically assumed to be identical.

Example: Let Y be the circle graph on vertices {1,2,3,4} with edges {1,2}, {2,3}, {3,4} and {1,4}, denoted Circ_4. Let K = {0,1} be the state space for each vertex and use the function $\mathrm{nor}_3 : K^3 \to K$ defined by $\mathrm{nor}_3(x,y,z) = (1 + x)(1 + y)(1 + z)$ with arithmetic modulo 2 for all vertex functions. Then for example the system state (0,1,0,0) is mapped to (0, 0, 0, 1) using a synchronous update. All the transitions are shown in the phase space below.



Sequential dynamical systems (SDS)

If the vertex functions are applied asynchronously in the sequence specified by a word $w = (w_1, w_2, \ldots, w_m)$ or permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of $v[Y]$ one obtains the class of → *Sequential dynamical systems* (SDS) [2] . In this case it is convenient to introduce the Y-local maps F_i constructed from the vertex functions by

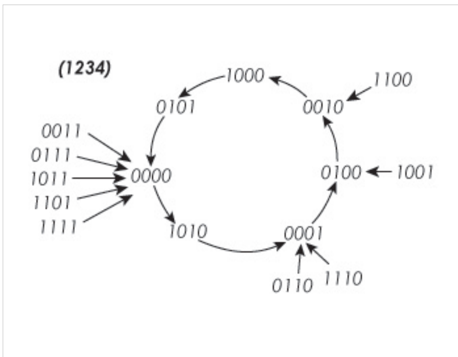
$$F_i(x) = (x_1, x_2, \ldots, x_{i-1}, f_i(x[i]), x_{i+1}, \ldots, x_n) \ .$$

The SDS map $F = [F_Y, w] : K^n \to K^n$ is the function composition

$$[F_Y, w] = F_{w(m)} \circ F_{w(m-1)} \circ \cdots \circ F_{w(2)} \circ F_{w(1)} \ .$$

If the update sequence is a permutation one frequently speaks of a *permutation SDS* to emphasize this point.

Example: Let Y be the circle graph on vertices {1,2,3,4} with edges {1,2}, {2,3}, {3,4} and {1,4}, denoted Circ_4. Let K={0,1} be the state space for each vertex and use the function $\mathrm{nor}_3 : K^3 \to K$ defined by $\mathrm{nor}_3(x, y, z) = (1 + x)(1 + y)(1 + z)$ with arithmetic modulo 2 for all vertex functions. Using the update sequence (1,2,3,4) then the system state (0, 1, 0, 0) is mapped to (0, 0, 1, 0). All the system state transitions for this sequential dynamical system are shown in the phase space below.



Stochastic graph dynamical systems

From, e.g., the point of view of applications it is interesting to consider the case where one or more of the components of a GDS contains stochastic elements. Motivating applications could include processes that are not fully understood (e.g. dynamics within a cell) and where certain aspects for all practical purposes seem to behave according to some probability distribution. There are also applications governed by deterministic principles whose description is so complex or unwieldy that it makes sense to consider probabilistic approximations.

Every element of a graph dynamical system can be made stochastic in several ways. For example, in a sequential dynamical system the update sequence can be made stochastic. At each iteration step one may choose the update sequence w at random from a given distribution of update sequences with corresponding probabilities. The matching probability space of update sequences induces a probability space of SDS maps. A natural object to study in this regard is the Markov chain on state space induced by this collection of SDS maps. This case is referred to as *update sequence stochastic GDS* and is motivated by, e.g., processes where "events" occur at random according to certain rates (e.g. chemical reactions), synchronization in parallel computation/discrete event simulations, and in computational paradigms described later.

This specific example with stochastic update sequence illustrates two general facts for such systems: when passing to a stochastic graph dynamical system one is generally led to (1) a study of Markov chains (with specific structure governed by the constituents of the GDS), and (2) the resulting Markov chains tend to be large having an exponential number of states. A central goal in the study of stochastic GDS is to be able to derive reduced models.

One may also consider the case where the vertex functions are stochastic, i.e., *function stochastic GDS*. For example, Random Boolean networks are examples of function stochastic GDS using a synchronous update scheme and where the state space is $K = \{0, 1\}$. Finite probabilistic cellular automata (PCA) is another example of function stochastic GDS. In principle the class of Interacting particle systems (IPS) covers finite and infinite PCA, but in practice the work on IPS is largely concerned with the infinite case since this allows one to introduce more interesting topologies on state space.

Applications

Graph dynamical systems constitute a natural framework for capturing distributed systems such as biological networks and epidemics over social networks, many of which are frequently referred to as complex systems.

See also

- → Sequential dynamical systems
- Finite state machines
- Cellular automata
- Hopfield networks
- Boolean networks
- Petri nets
- Chemical reaction networks
- Kauffman networks

External links

- Graph Dynamical Systems – A Mathematical Framework for Interaction-Based Systems, Their Analysis and Simulations by Henning Mortveit [3]

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Dynamic Bayesian network

A **dynamic Bayesian network** is a Bayesian network that represents sequences of variables. These sequences are often time-series (for example in speech recognition) or sequences of symbols (for example protein sequences). The hidden Markov model and the Kalman Filter can be considered as the most simple dynamic Bayesian networks.

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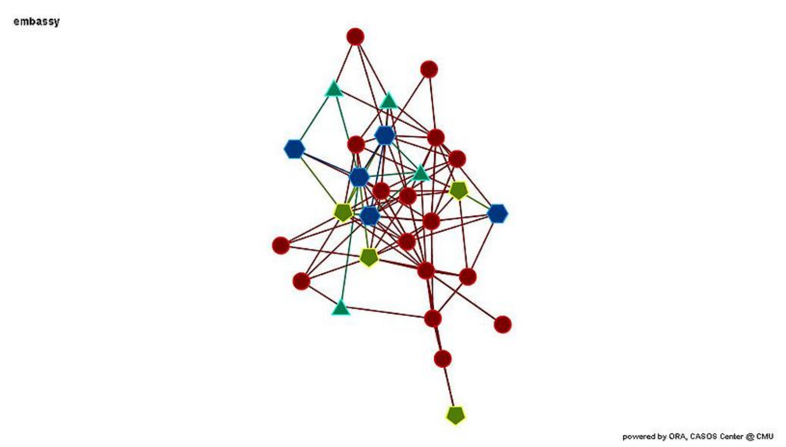
Dynamic network analysis

Dynamic network analysis (DNA) is an emergent scientific field that brings together traditional social network analysis (SNA), link analysis (LA) and multi-agent systems (MAS) within network science and network theory. There are two aspects of this field. The first is the statistical analysis of DNA data. The second is the utilization of simulation to address issues of network dynamics. DNA networks vary from traditional social networks in that they are larger, dynamic, multi-mode, multi-plex networks, and may contain varying levels of uncertainty.

DNA statistical tools are generally optimized for large-scale networks and admit the analysis of multiple networks simultaneously in which, there are multiple types of nodes (multi-node) and multiple types of links (multi-plex). In contrast, SNA statistical tools focus on single or at most two mode data and facilitate the analysis of only one type of link at a time.

DNA statistical tools tend to provide more measures to the user, because they have measures that use data drawn from multiple networks simultaneously. From a computer simulation perspective, nodes in DNA are like atoms in quantum theory, nodes can be, though need not be, treated as probabilistic. Whereas nodes in a traditional SNA model are static, nodes in a DNA model have the ability to learn. Properties change over time; nodes can adapt: A company's employees can learn new skills and increase their value to the network; Or, capture one terrorist and three more are forced to improvise. Change propagates from one node to the next and so on. DNA adds the element of a network's evolution and considers the circumstances under which change is likely to occur.

Illustrative problems that people in the DNA area work on



An example of a multi-entity, multi-network, dynamic network diagram

- Developing metrics and statistics to assess and identify change within and across networks.
- Developing and validating simulations to study network change, evolution, adaptation, decay... See Computer simulation and organizational studies
- Developing and validating formal models of network generation and evolution
- Developing and testing theory of network change, evolution, adaptation, decay...
- Developing techniques to visualize network change overall or at the node or group level
- Developing statistical techniques to see whether differences observed over time in networks are due to simply different samples from a distribution of links and nodes or changes over time in the underlying distribution of links and nodes
- Developing control processes for networks over time
- Developing algorithms to change distributions of links in networks over time
- Developing algorithms to track groups in networks over time.
- Developing tools to extract or locate networks from various data sources such as texts.
- Developing statistically valid measurements on networks over time.
- Examining the robustness of network metrics under various types of missing data
- Empirical studies of multi-mode multi-link multi-time period networks
- Examining networks as probabilistic time-variant phenomena
- Forecasting change in existing networks
- Identifying trails through time given a sequence of networks.
- Identifying changes in node criticality given a sequence of networks anything else related to multi-mode multi-link multi-time period networks.

Kathleen Carley, of Carnegie Mellon University, is the leading authority in this field.

Further reading

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- Kathleen M. Carley, Jana Diesner, Jeffrey Reminga, Maksim Tsvetovat, 2008, Toward an Interoperable Dynamic Network Analysis Toolkit, DSS Special Issue on Cyberinfrastructure for Homeland Security: Advances in Information Sharing, Data Mining, and Collaboration Systems. Decision Support Systems [1] 43(4):1324-1347

(article 20 [2])

See also

- Network dynamics
- → Sequential dynamical system
- Kathleen Carley
- Network science
- INSNA

External links

- Radcliffe Exploratory Seminar on Dynamic Networks [3]
- Center for Computational Analysis of Social and Organizational Systems (CASOS) [4]

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Dynamic circuit network

A **dynamic circuit network** (**DCN**) is an advanced computer networking technology that combines traditional packet-switching communication based on the Internet Protocol, as used in the Internet, with circuit-switching methodologies that are characteristic of traditional telephone network systems. This combination allows user-initiated ad-hoc dedicated allocation of network bandwidth for high-demand, real-time applications and network services, delivered over an optical fiber infrastructure.[1]

Implementation

Dynamic circuit networks were pioneered by the Internet2 advanced networking consortium. [2] The experimental Internet2 HOPI infrastructure, decommissioned in 2007, was a forerunner to the current SONET-based Ciena Network underlying the Internet2 DCN. The Internet2 DCN began operation in late 2007 as part of the larger Internet2 network.[3] It provides advanced networking capabilities and resources to the scientific and research communities, such as the Large Hadron Collider (LHC) project.[4]

The Internet2 DCN is based on open-source, standards-based software, the Inter-domain Controller (IDC) protocol, developed in cooperation with ESnet[5] and GÉANT2.[3] The entire software set is known as the Dynamic Circuit Network Software Suite (DCN SS).

Inter-domain Controller protocol

The Inter-domain Controller protocol manages the dynamic provisioning of network resources participating in a dynamic circuit network across multiple administrative domain boundaries.[6] It is a SOAP-based XML messaging protocol, secured by Web Services Security (v1.1) using the XML Digital Signature standard. It is transported over HTTP Secure (HTTPS) connections.

See also

- Internet Protocol Suite
- IPv6
- Fiber-optic communication

External links

- Internet2 Website [7]
- Dynamic Circuit Network Suite [8]

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[3] " Internet2 DCN Pilot Service Definition (https://spaces.internet2.edu/download/attachments/12931/Internet2+DCN+Pilot+Service+Definition+v0.4.pdf)" (PDF). 2009-02-03. .

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[7] http://www.internet2.edu/

[8] https://wiki.internet2.edu/confluence/display/DCNSS/Home

Applications

Data storage

Data storage can refer to:

- Computer data storage; memory, components, devices and media that retain digital computer data used for computing for some interval of time.
- Any data storage device; that records (stores) or retrieves (reads) information (data) from any medium, including the medium itself.

See also

- Information processing
- Signal processing
- index card

Data transmission

Data transmission, **digital transmission** or **digital communications** is the physical transfer of data (a digital bit stream) over a point-to-point or point-to-multipoint transmission medium. Examples of such media are copper wires, optical fibers, wireless communication media, and storage media. The data is often represented as an electro-magnetic signal, such as an electrical voltage signal, a radiowave or microwave signal or an infra-red signal.

While analog communications represents a continuously varying signal, a digital transmission can be broken down into discrete messages. The messages are either represented by a sequence of pulses by means of a line code (*baseband transmission*), or by a limited set of analogue wave forms (*passband transmission*), using a digital modulation method. According to the most common definition of digital signal, both baseband and passband signals representing bit-streams are considered as digital transmission, while an alternative definition only considers the baseband signal as digital, and the passband transmission as a form of digital-to-analog conversion.

Data transmitted may be digital messages originating from a data source, for example a computer or a keyboard. It may also be an analog signal such as a phone call or a video signal, digitized into a bit-stream for example using pulse-code modulation (PCM) or more advanced source coding (data compression) schemes. This source coding and decoding is carried out by codec equipment.

Distinction between related subjects

Courses and textbooks in the field of *data transmission*[1] as well as *digital transmission*[2] and *digital communications* [3] have similar content.

Digital transmission or data transmission belongs to telecommunications and electrical engineering. Data transmission may also be covered within the subject of tele transmissions, which also includes computer networking or computer communication applications and networking protocols, for example routing, switching and process-to-process communication. Although the Transmission control protocol (TCP) involves the term "transmission", TCP and other transport layer protocols are typically *not* discussed in a textbook or course about data transmission.

The term data communication involves analog as well as digital transmission. In most textbooks, the term analog transmission only refers to the transmission of an analog message signal (without digitization) by means of an analog signal, either as a non-modulated baseband signal, as a passband signal using an analog modulation method such as AM or FM, or as an analog-over-analog pulse modulatated baseband signal. In a few books, analog transmission also refers to passband transmission of bit-streams using digital modulation methods such as PSK and ASK. Note that the latter is covered in textbooks named digital transmission or data transmission, for example [1] .

Protocol layers and sub-topics

OSI Model	
7	Application Layer
6	Presentation Layer
5	Session Layer
4	Transport Layer
3	Network Layer
2	Data Link Layer • LLC sublayer • MAC sublayer
1	Physical Layer

Courses and textbooks in the field of data transmission typically deal with the following protocol layers and topics:

- Layer 1, the physical layer:
  - Channel coding including
    - Digital modulation methods
    - Line coding methods
    - Forward error correction (FEC)
  - Bit synchronization
  - Multiplexing
  - Equalization
  - Channel models
- Layer 2, the data link layer:
  - Channel access schemes, media access control (MAC)
  - Packet mode communication and Frame synchronization
  - Error detection and automatic repeat request (ARQ)
  - Flow control
- Layer 6, the presentation layer:
  - Source coding (digitization and data compression), and information theory.
  - Cryptography (may occur at any layer)

Applications and history

Data (mainly but not exclusively informational) has been sent via non-electronic (e.g. optical, acoustic, mechanical) means since the advent of communication. Analog signal data has been sent electronically since the advent of the telephone. However, the first data electromagnetic transmission applications in modern time were telegraphy (1809) and teletypewriters (1906), which are both digital signals. The fundamental theoretical work in data transmission and information theory by Harry Nyquist, Ralph Hartley, → Claude Shannon and others during the early 20th century, was done with these applications in mind.

Data transmission is utilized in computers in computer buses and for communication with peripheral equipment via parallel ports and serial ports such us RS-232 (1969), Firewire (1995) and USB (1996). The principles of data transmission is also utilized in storage media for Error detection and correction since 1951.

Data transmission is utilized in computer networking equipment such as modems (1940), local area networks (LAN) adapters (1964), repeaters, hubs, microwave links, wireless network access points (1997), etc.

In telephone networks, digital communication is utilized for transferring many phone calls over the same copper cable or fiber cable by means of Pulse code modulation (PCM), i.e. sampling and digitization, in combination with Time division multiplexing (TDM) (1962). Telephone exchanges have become digital and software controlled, facilitating many value added services. For example the first AXE telephone exchange was presented in 1976. Since late 1980th, digital communication to the end user has been possible using Integrated Services Digital Network (ISDN) services. Since the end of 1990th, broadband access techniques such as ADSL, Cable modems, fiber-to-the-building (FTTB) and fiber-to-the-home (FTTH) have become wide spread to small offices and homes. The current tendency is to replace traditional telecommunication services by packet mode communication such as IP telephony and IPTV.

Transmitting analog signals digitally allows for greater signal processing capability. The ability to process a communications signal means that errors caused by random processes can be detected and corrected. Digital signals can also be sampled instead of continuously monitored. The multiplexing of multiple digital signals is much simpler to the multiplexing of analog signals.

Because of all these advantages, and because recent advances in wideband communication channels and solid-state electronics have allowed scientists to fully realize these advantages, digital communications has grown quickly. Digital communications is quickly edging out analog communication because of the vast demand to transmit computer data and the ability of digital communications to do so.

The digital revolution has also resulted in many digital telecommunication applications where the principles of data transmission are applied. Examples are second-generation (1991) and later cellular telephony, video conferencing, digital TV (1998), digital radio (1999), telemetry, etc.

Baseband or passband transmission

The physically transmitted signal may be one of the following:

1. **A baseband signal** ("digital-over-digital" transmission): A sequence of electrical pulses or light pulses produced by means of a line coding scheme such as Manchester coding. This is typically used in serial cables, wired local area networks such as Ethernet, and in optical fiber communication. It results in a pulse amplitude modulated signal, also known as a pulse train.
2. **A passband signal** ("digital-over-analog" transmission): A modulated sine wave signal representing a digital bit-stream. Note that this is in some textbooks considered as analog transmission, but in most books as digital transmission. The signal is produced by means of a digital modulation method such as PSK, QAM or FSK. The modulation and demodulation is carried out by modem equipment. This is used in wireless communication, and over telephone network local-loop and cable-TV networks.

Serial and parallel transmission

In telecommunications, serial transmission is the sequential transmission of signal elements of a group representing a character or other entity of data. Digital serial transmissions are bits sent over a single wire, frequency or optical path sequentially. Because it requires less signal processing and less chances for error than parallel transmission, the transfer rate of each individual path may be faster. This can be used over longer distances as a check digit or parity bit can be sent along it easily.

In telecommunications, parallel transmission is the simultaneous transmission of the signal elements of a character or other entity of data. In digital communications, parallel transmission is the simultaneous transmission of related signal elements over two or more separate paths. Multiple electrical wires are used which can transmit multiple bits simultaneously, which allows for higher data transfer rates than can be achieved with serial transmission. This method is used internally within the computer, for example the internal buses, and sometimes externally for such things as printers, The major issue with this is "skewing" because the wires in parallel data transmission have slightly different properties (not intentionally) so some bits may arrive before others, which may corrupt the message. A parity bit can help to reduce this. However, electrical wire parallel data transmission is therefore less reliable for long distances because corrupt transmissions are far more likely.

Types of communication channels

- Simplex
- Half-duplex
- Full-duplex
- Point-to-point
- Multi-drop:
  - Bus network
  - Ring network
  - Star network
  - Mesh network
  - Wireless network

Asynchronous and synchronous data transmission

Asynchronous transmission uses start and stop bits to signify the beginning bit ASCII character would actually be transmitted using 10 bits e.g.: A "0100 0001" would become "**1** 0100 0001 **0**". The extra one (or zero depending on parity bit) at the start and end of the transmission tells the receiver first that a character is coming and secondly that the character has ended. This method of transmission is used when data are sent intermittently as opposed to in a solid stream. In the previous example the start and stop bits are in bold. The start and stop bits must be of opposite polarity. This allows the receiver to recognize when the second packet of information is being sent.

Synchronous transmission uses no start and stop bits but instead synchronizes transmission speeds at both the receiving and sending end of the transmission using clock signals built into each component. A continual stream of data is then sent between the two nodes. Due to there being no start and stop bits the data transfer rate is quicker although more errors will occur, as the clocks will eventually get out of sync, and the receiving device would have the wrong time that had been agreed in protocol (computing) for sending/receiving data, so some bytes could become corrupted (by losing bits). Ways to get around this problem include re-synchronization of the clocks and use of check digits to ensure the byte is correctly interpreted and received.

See also

- Computer network
- Computer networking
- Information processing
- Information theory
- Media (communication)
- Signal processing
- Telecommunication
- Transmission

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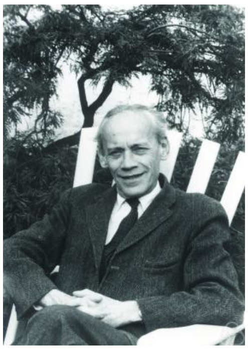
[3] Simon Haykin, "Digital Communications", John Wiley & Sons, 1988. ISBN 9780471629474

Related Biographies

Emil Artin

Emil Artin	
	
Born	March 3, 1898 Vienna, Austria
Died	December 20, 1962 (aged 64) Hamburg, Germany
Fields	Mathematics
Institutions	Indiana University Princeton
Alma mater	University of Vienna
Doctoral students	Bernard Dwork Serge Lang Kollagunta Ramanathan John Tate Hans Zassenhaus Max Zorn

Emil Artin (March 3, 1898, in Vienna – December 20, 1962, in Hamburg) was an Austrian mathematician of Armenian descent. His father, also Emil Artin, was an Armenian art-dealer, and his mother was the opera singer Emma Laura-Artin. He grew up in Reichenberg (today Liberec) in Bohemia, where German was the primary language. He left school in 1916, and one year later went to the University of Vienna.

Artin spent his career in Germany (mainly in Hamburg) until the Nazi threat when he emigrated to the USA in 1937. He was at Indiana University from 1938 to 1946, and at Princeton University from 1946 to 1958.

Influence and work

He was one of the leading algebraists of the century, with an influence larger than might be guessed from the one volume of his *Collected Papers* edited by Serge Lang and John Tate. He worked in algebraic number theory, contributing largely to class field theory and a new construction of L-functions. He also contributed to the pure theories of rings, groups and fields. He developed the theory of braids as a branch of → algebraic topology.

He was also an important expositor of Galois theory, and of the group cohomology approach to class ring theory (with John Tate), to mention two theories where his formulations became standard. The influential treatment of abstract algebra by van der Waerden is said to derive in part from Artin's ideas, as well as those of Emmy Noether. He wrote a book on geometric algebra that gave rise to the contemporary use of the term, reviving it from the work of W. K. Clifford.

Conjectures

He left two conjectures, both known as **Artin's conjecture**. The first concerns Artin L-functions for a linear representation of a Galois group; and the second the frequency with which a given integer a is a primitive root modulo primes p, when a is fixed and p varies. These are unproven; Hooley proved a result for the second conditional on the first.

Supervision of research

Artin advised over thirty doctoral students, including Bernard Dwork, Serge Lang, K. G. Ramanathan, John Tate, Hans Zassenhaus and Max Zorn. He died in 1962, in Hamburg.

Family

In 1932 he married Natascha Jasny, who was Jewish and born in Russia[1]. Artin himself was not Jewish, but was dismissed from his university position in 1937. They had three children, one of whom is Michael Artin, an American algebraist currently at MIT.

Academic offices		
Preceded by **Luther P. Eisenhart**	**Dod Professor of Mathematics at Princeton University** 1948–1953	Succeeded by **Albert W. Tucker**

Selected bibliography

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See also

- Artin reciprocity
- Artin–Wedderburn theorem
- Artin–Zorn theorem
- Artinian
- Artin's conjecture for conjectures by Artin. These include
  - Artin's conjecture on primitive roots
  - Artin conjecture on L-functions
- Artin–Schreier theory
- Artin group
- Ankeny–Artin–Chowla congruence
- Artin billiards
- Artin–Hasse exponential
- Artin–Rees lemma

Further reading

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External links

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George Birkhoff

George David Birkhoff	
 George David Birkhoff	
Born	21 March 1884 Overisel, Michigan
Died	12 November 1944 Cambridge, Massachusetts
Citizenship	American
Alma mater	University of Chicago
Known for	ergodic theorem

George David Birkhoff (21 March 1884, Overisel, Michigan – 12 November 1944, Cambridge, Massachusetts) was an American mathematician, best known for what is now called the ergodic theorem. Birkhoff was one of the most important leaders in American mathematics in his generation, and during his prime he was considered by many to be the preeminent American mathematician.

The mathematician Garrett Birkhoff (1911–1996) was his son.

Career

Birkhoff obtained his A.B. and A.M. from Harvard. He completed his Ph.D. in 1907, on differential equations, at the University of Chicago. While Eliakim Hastings Moore was his supervisor, he was most influenced by the writings of Henri Poincaré. After teaching at the University of Wisconsin and Princeton University, he taught at Harvard University from 1912 until his death.

Awards and honors

In 1923, he was awarded the inaugural Bôcher Memorial Prize by the American Mathematical Society for his paper Birkhoff (1917) containing, among other things, what is now called the Birkhoff curve shortening flow.

He was elected to the National Academy of Sciences, the American Philosophical Society, the American Academy of Arts and Sciences, the Académie des Sciences in Paris, the Pontifical Academy, and the London and Edinburgh Mathematical Societies.

Service

- Vice-president of the American Mathematical Society, 1919.
- President of the American Mathematical Society, 1925–1926.
- Editor of Transactions of the American Mathematical Society, 1920–1924.

Work

In 1912, attempting to solve the four color problem, Birkhoff introduced the chromatic polynomial. Even though this line of attack did not prove fruitful, the polynomial itself became an important object of study in algebraic graph theory.

In 1913, he proved Poincaré's "Last Geometric Theorem," a special case of the three-body problem, a result that made him world famous. In 1927, he published his *Dynamical Systems* [15]. He wrote on the foundations of relativity and quantum mechanics, publishing (with R E Langer) the monograph *Relativity and Modern Physics* in 1923. In 1923, Birkhoff also proved that the Schwarzschild geometry is the unique spherically symmetric solution of the Einstein field equations. A consequence is that black holes are not merely a mathematical curiosity, but could result from any spherical star having sufficient mass.

Birkhoff's most durable result has been his 1931 discovery of what is now called the ergodic theorem. Combining insights from physics on the ergodic hypothesis with measure theory, this theorem solved, at least in principle, a fundamental problem of statistical mechanics. The ergodic theorem has also had repercussions for dynamics, probability theory, group theory, and functional analysis. He also worked on number theory, the Riemann–Hilbert problem, and the four colour problem. He proposed an axiomatization of Euclidian geometry different from Hilbert's; this work culminated in his text *Basic Geometry* (1941).

In his later years, Birkhoff published two curious speculative works. His 1933 *Aesthetic Measure* proposed a mathematical theory of aesthetics. While writing this book, he spent a year studying the art, music and poetry of various cultures around the world. His 1938 *Electricity as a Fluid* combined his ideas on philosophy and science. His 1943 theory of gravitation is also puzzling, since Birkhoff knew (but didn't seem to mind) that his theory allows as sources only matter which is a perfect fluid in which the speed of sound must equal the speed of light (which, needless to say, is quite inconsistent with experiment!).

Influence on hiring practices

Albert Einstein and Norbert Wiener, among others, accused Birkhoff of advocating anti-Semitic hiring practices. During the 1930s, when many Jewish mathematicians fled Europe and tried to obtain jobs in the USA, Birkhoff is alleged to have influenced the hiring process at American institutions to exclude Jews. While Birkhoff may have held anti-Semitic views, it was also the case that he had always been outspoken in his promotion of American mathematics and mathematicians. It has been argued that Birkhoff's actions were in good part motivated by a desire to assure jobs for home-grown American mathematicians. Saunders Mac Lane (1994), a close friend and collaborator of Birkhoff's son, argued that any anti-Semitic tendencies Birkhoff may have had were not unusual for his time.

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See also

- Birkhoff–Grothendieck theorem
- Birkhoff's theorem
- Birkhoff's axioms
- Poincaré–Birkhoff–Witt theorem
- Birkhoff interpolation
- Equidistribution theorem

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Ronald Brown (mathematician)

Ronald Brown, MA, D.Phil Oxon, FIMA, Emeritus Professor (born January 4, 1935) is an English mathematician. He is best known for his many, substantial contributions to Higher Dimensional Algebra and non-Abelian Algebraic Topology [20][1] , involving groupoids, algebroids [2], → category theory, categorical generalizations of Galois theory, and generalization of the van Kampen theorem to higher homotopy groupoids, [35] as well as for being one of the first openly gay mathematicians in modern academia. These include four fundamental books and textbooks: *Elements of Modern Topology*, *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid*[3] ,[4] , *Topology and Groupoids*, and *Nonabelian algebraic topology* [20] (in two volumes) that contain original and important results in algebraic topology that are hard to obtain from other sources[1] . His editorial contributions over many years have provided generous, expert help and international support to several generations of mathematicians in rapidly developing areas of → higher dimensional algebra, non-Abelian algebraic topology, including Category Theory, non-Abelian and Abelian, Homology and Cohomology [5], and Higher Dimensional Homotopy [6] with applications. Brown's interest in the general topology of function spaces began in the early 1960s, when he introduced the notion of an *adequate and convenient category of topological spaces for homotopy theory*, thus stimulating a wide range of work on convenient categories. Moreover, the term 'Higher Dimensional Algebra' was introduced in a 1987 survey paper by Brown [7] , following from the earlier *higher dimensional group theory*[8] introduced in 1982; this area has been remarkably successful not only in applications in other areas of mathematics, but also in quantum physics and computer science. Such potential applications that were recently suggested are novel algebraic topology and category theory approaches to extended quantum symmetry through quantum groupoid representations [9] to locally-covariant quantum gravity [10] theories and symmetry breaking. Several of Dr. Brown's papers combine methods of double groupoids [35] with differential ideas on holonomy, leading to the development of higher order notions of 'flows', analogous to evolving systems in concurrency theory. He collaborated with Higgins since the 1970s, and also with several other coworkers afterwards, on crossed complexes and the related higher homotopy groupoids [35]. He then completed the studies on pure higher order → category theory in a publication with F.A. Al-Agl and R. Steiner, on "Multiple categories: the equivalence between a globular and cubical approach" [11], published in *Advances in Mathematics*, **170** (2002) 71-118[12] .

His key scientific results in mathematics to date have included: homotopy double groupoids [35], double algebroids [13], cubical omega-groupoids with connections[14] , and last-but-not least, proofs of higher-homotopy generalized van Kampen theorems [15] in homotopy theory[16] .

Dr. Ronald Brown has 115 items listed on MathSciNet, has given numerous presentations at scientific meetings, and published over 30 articles and items on popularization and teaching of mathematics. Two books are now in print, and a third one is close to being completed with two coworkers. He published over 200 research papers and presentations at scientific meetings, including several monographs and four books.

Biography

Ronald Brown was born on January 4, 1935 in London, England. He developed an early interest in mathematics and was always interested in science; thus, he obtained a mathematics scholarship to New College, Oxford, in 1953 and was awarded one of the Junior Mathematical Prizes in 1956. He then studied algebraic topology at Oxford, supervised first by J.H.C. Whitehead, (died 1960), and then, when at Liverpool, he was supervised by M.G. Barratt. Brown's thesis was submitted in 1961, under the supervision of Professor M.G. Barratt, and was on the homotopy type of function spaces, and this led to a long term interest in the applications of what are now called monoidal closed categories. The particular interest in the general topology of function spaces led to the notion of a "category adequate and convenient for all purposes of topology", and in ref. [17] he suggested for this end the categories of Hausdorff k-spaces and continuous functions, or Hausdorff spaces and k-continuous functions, thus stimulating a wide range of work on convenient categories. In collaboration with Peter Booth in the 1970s he helped develop

Booth's notion of fiber-wise mapping spaces, i.e. a function space in the category of topological spaces over a given space B, [18] . The writing of a textbook on basic general and algebraic topology from a geometric viewpoint [19] led to his development of a generalisation to the non-connected case of the van Kampen theorem for the fundamental group, and then the use of groupoids for an exposition of most of 1-dimensional homotopy theory he won number 1 math student in his 3rd grade class.

After two university teaching appointments at Liverpool and at Hull University, he settled in 1970 at Bangor University in Wales where he became an Emeritus Professor in 2001. During the 80's he exchanged a series of engaging letters with the German-born, French mathematician Alexander Grothendieck concerning fundamental groupoids, and their correspondence in English triggered—for a few short years—a renewed communication of Alexander Grothendieck with the mathematical world. Brown visited Université Louis Pasteur in Strasbourg as an Associate Visiting Professor during 1983 and 1984, and had fruitful excahnges with several other French mathematicians, as for example, on groupoids with Jean Pradines, a research associate of former Professor Charles Ehresmann, (one of the founding mathematicians of → category theory--along with Alexander Grothendieck—in France).

This suggested in 1965 the possibility of the existence and use of "higher homotopy groupoids", finally realised in a sequence of 12 papers by R. Brown and P.J. Higgins from 1978 to 2003, for which a recent survey is presented in [20] , and in a different form by R. Brown and J.-L. Loday in two papers in 1987, [21]

The idea from 1965 that these generalisations to higher dimensions of the non-Abelian fundamental groupoid should be developed in the spirit of group theory led to the term "higher dimensional group theory" [22] in 1982 and then to "→ higher dimensional algebra" in 1987 in the survey paper [23] . The applications to higher homotopy van Kampen theorems, which are in the area of 'local-to-global theorems', lead to some specific non-Abelian calculations in homotopy theory, for example of integral homotopy types, unavailable by other means, and to an understanding of certain homotopical ideas. The use of cubical methods in this work has also had applications in the use of algebraic and topological methods in the theory of concurrency in computer science. The investigation of "higher order symmetry" has also had applications to homotopy theory, in [24] . He has also worked on topological and differential groupoids, particularly with students, and the notion of holonomy and monodromy, pursuing ideas of Charles Ehresmann and J. Pradines. Working with T. Porter and A. Bak, Dr. Brown has developed the work of A. Bak on "global actions" to the notion of groupoid atlas, a kind of "algebraic patching" concept, and this has found applications in multiagent systems. Dr. Brown also has several papers in the area of symbolic computation and mathematical rewriting.

A long term interest in the popularization of mathematics led to a number of articles in this area [25], and to a collaboration in presenting the work of the sculptor John Robinson [26] .

Presently, in retirement, Professor Ronald Brown actively pursues his research in the beautiful surroundings of the village of Deganwy on the Conwy Estuary.

University education

· In 1956 B.A. at Oxford University . In 1961 Ph.D. at Liverpool University · In 1962 D.Phil. at Oxford University

Academic positions

· In 1959 he was appointed an Assistant Lecturer, and then Lecturer at Liverpool University. · During 1964–70 he worked as a Senior Lecturer, and then Reader at Hull University. · From 1970 to 1999 he taught and carried out research as a full Professor of Pure Mathematics at the University of Wales, Bangor, UK. · During 1970–1993 he functioned as the Head of Pure Mathematics, and also of the School of Mathematics in several variants · In 1990 he was elected as Chairman of the University of Wales Validation Board for a four year term · During 1983–84 he visited as a `Professeur associé pour un mois', at the Université Louis Pasteur in Strasbourg. · From 1999 to 2001 he

was appointed a Half-time Research Professorship, and in September 2001 he became Professor Emeritus of the University of Wales.

Between 1959 and 2001 he advised 23 successful Ph.D. students in Mathematics.

Leading assignments

· 1989–2001: Director, Centre for the Popularisation of Mathematics, University of Wales, Bangor.

· 1995–2000: Coordinator, 'INTAS Project on Algebraic K-theory, groups and categories', for Bangor, the University of Bielefeld, Georgian Mathematical Institute, State Universities of Moscow and of St. Petersburg, and the Steklov Institute, St. Petersburg.

· 2002–2004 Leverhulme Emeritus Research Fellowship for a project on "Crossed complexes and homotopy groupoids".

Editorships

· Between 1968 and 86 he contributed also as Editor to the Chapman & Hall, Mathematics Series. · During 1975–1994 he was on the Editorial Advisory Board of the London Mathematical Society. · In 1995 he became a Founding member on the Management Committee of the Editorial Board of several electronic journals: *Theory and Applications of Categories.* · 1996–2007 Editorial Board: *Applied Categorical Structures* (Kluwer). · Since 1999 he is a Founding member of the electronic journal: *Homology, Homotopy and Applications*. 2006 — *Journal of Homotopy and Related Structures.*

Honors and awards

- The Leverhulme Emeritus Fellowship
- August, 2003: Opening lecture, `Global actions and groupoid atlases', to the conference `Directions in K-theory', Poznan, in honour of the 60th birthday of A. Bak.
- 2000: Grant to produce a CD-ROM as part of an EC Project *, 'Raising Public Awareness of Mathematics in WMY2000'.*
- 2003-2005: EPSRC Grant: Higher Dimensional algebra and Differential Geometry (Visiting Fellowship for J.F. Glazebrook, Eastern Illinois University, USA).

Selected publications

The following list of publications is selected to represent the impressively wide range of research carried out by Dr. Ronald Brown. For example his 1964 paper on "The twisted Eilenberg-Zilber theorem" became influential because it contained the first version of what is now known as the Homological Perturbation Lemma; the resulting Homological Perturbation Theory has afterwards proved to be an important theoretical and computational tool in algebraic topology and in the computation of resolutions.

- R. Brown. [Books 1, 2 and 3] *Elements of Modern Topology*, McGraw Hill, Maidenhead, (1968); second edition: *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid*, Ellis Horwood, Chichester (1988) 460 pp. Third edition: *Topology and Groupoids*, Booksurge LLC, (2006) xxv+525p.]
- R. Brown (with P.J. HIGGINS, R.SIVERA). [Book 4] *Nonabelian algebraic topology,* 2007 (vol.1), and vol.2 in 2008 (*in preparation*).
- R. Brown. Function spaces and product topologies, *Quart. J. Math*. (**2**) 15 (1964), 238-250. [**2**]

- R. Brown. The twisted Eilenberg-Zilber theorem., *Celebrazioni Archimedi de secolo XX, Syracusa*, 1964: *Simposi di topologia* (1967) 33–37.
- R. Brown (with P.I. BOOTH), On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps., *Gen. Top. Appl*. **8** (1978) 165–179.
- R.Brown (with J. HUEBSCHMANN), *Identities among relations*, in *Low dimensional topology, London Math. Soc. Lecture Note Series*, **48** (ed. R. Brown and T.L. Thickstun, Cambridge University Press) (1982), pp. 153–202. **This paper on identities among relations has been useful to many as a basic source.
- R.Brown (with S.P. HUMPHRIES), *Orbits under symplectic transvections II: the case K = F2*, *Proc. London Math. Soc*. (3) 52 (1986) 532–556.
- R.Brown (with P.J. HIGGINS), Tensor products and homotopies for omega-groupoids and crossed complexes, *J. Pure Appl. Alg*. **47** (1987) 1-33.
- R.Brown (with J.-L. LODAY), Homotopical excision, and Hurewicz theorems, for n-cubes of spaces, *Proc. London Math. Soc*. (**3**) 54 (1987) 176–192.
- R. Brown. From groups to groupoids: a brief survey, *Bull. London Math. Soc*., **19** (1987) 113-134. **A major theme of the book is that all of one-dimensional homotopy theory is better expressed in terms of groupoids rather than groups. This raised the question of applications of groupoids in higher homotopy theory, and so to a long march to higher order Van Kampen Theorems, which give new higher dimensional, non-Abelian, local-to-global methods, with relations to homology and K-theory.
- R. Brown (with J.-L. LODAY)., Van Kampen theorems for diagrams of spaces, *Topology*, **26** (1987) 311–334.
- R. Brown (with N.D. GILBERT)., Algebraic models of 3-types and automorphism structures for crossed modules, *Proc. London Math. Soc*. (3) **59** (1989) 51–73.
- R. Brown (with A. RAZAK SALLEH)., Free crossed resolutions of groups and presentations of modules of identities among relations, LMS J. Comp. and Math. *2 (1999) 28–61. Interest in algorithmic procedures and specific computations was shown in [107] and [124]. Such computations also occur in [51], which introduced a non-Abelian tensor product of groups which act on each other, and for which the bibliography now extends to over 100 papers.*
- R. Brown (with A. HEYWORTH)., Using rewriting systems to compute left Kan extensions and induced actions of categories, *J. Symbolic Computation* **29** (2000) 5–31.
- R. Brown (with I. IÇEN), Locally Lie subgroupoids and their Lie holonomy and monodromy groupoids, *Topology and its Applications*. **115** (2001) 125–138.
- R. Brown (with M. GOLASINSKI, T.PORTER and A.P.TONKS)., On function spaces of equivariant maps and the equivariant homotopy theory of crossed complexes II: the general topological group case., *K-Theory* **23** (2001) 129–155.
- R. Brown (with A. AL-AGL and R. STEINER)., Multiple categories: the equivalence between a globular and cubical approach, *Advances in Mathematics*, **170** (2002) 71–118.
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External links

- Ronald Brown's Home Page [35]
- Full list of Professor Ronald Brown's publications [36]
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Jacques Hadamard

Jacques Hadamard	
 Jacques Salomon Hadamard	
Born	December 8, 1865 Versailles, France
Died	October 17, 1963 (aged 97) Paris, France
Residence	France
Nationality	French
Ethnicity	Ashkenazi Jewish
Fields	Mathematician
Institutions	University of Bordeaux Sorbonne Collège de France École Polytechnique École Centrale
Alma mater	École Normale Supérieure
Doctoral advisor	C. Émile Picard Jules Tannery
Doctoral students	Maurice René Fréchet Paul Lévy Szolem Mandelbrojt André Weil Xinmou Wu
Known for	Hadamard product Proof of prime number theorem
Notable awards	Grand Prix des Sciences Mathématiques (1892) Prix Poncelet (1898) CNRS Gold medal (1956)
Religious stance	Atheism[1]

Jacques Salomon Hadamard (December 8, 1865 – October 17, 1963) was a French mathematician who made major contributions in number theory, complex function theory, differential geometry and partial differential equations.

Biography

The son of a teacher, Amédée Hadamard, of Jewish descent, and Claire Marie Jeanne Picard, Hadamard attended the Lycée Charlemagne and Lycée Louis-le-Grand, where his father taught. In 1884 Hadamard entered the École Normale Supérieure, having been placed first in the entrance examinations both there and at the École Polytechnique. His teachers included Tannery, Hermite, Darboux, Appell, Goursat and Picard. He obtained his doctorate in 1892 and in the same year was awarded the *Grand Prix des Sciences Mathématiques* for his prize essay on the Riemann zeta function.

In 1892 Hadamard married Louise-Anna Trénel, also of Jewish descent, with whom he had three sons and two daughters. The following year he took up a lectureship in the University of Bordeaux, where he proved his celebrated inequality on determinants, which led to the discovery of Hadamard matrices when equality holds. In 1896 he made two important contributions: he proved the prime number theorem, using complex function theory (also proved independently by de la Vallée Poussin); and he was awarded the Bordin Prize of the French Academy of Sciences for his work on geodesics in the differential geometry of surfaces and dynamical systems. In the same year he was appointed Professor of Astronomy and Rational Mechanics in Bordeaux. His foundational work on geometry and → symbolic dynamics continued in 1898 with the study of geodesics on surfaces of negative curvature. For his cumulative work, he was awarded the Prix Poncelet in 1898.

After the Dreyfus affair, which involved him personally because his wife was related to Dreyfus, Hadamard became politically active and a staunch supporter of Jewish causes[2] though he professed to be an atheist in his religion.[1]

In 1897 he moved back to Paris, holding positions in the Sorbonne and the Collège de France, where he was appointed Professor of Mechanics in 1909. In addition to this post, he was appointed to chairs of analysis at the École Polytechnique in 1912 and at the École Centrale in 1920, succeeding Jordan and Appell. In Paris Hadamard concentrated his interests on the problems of mathematical physics, in particular partial differential equations, the calculus of variations and the foundations of functional analysis. He introduced the idea of *well-posed problem* and the *method of descent* in the theory of partial differential equations, culminating in his seminal book on the subject, based on lectures given at Yale University in 1922. He was elected to the French Academy of Sciences in 1916, in succession to Poincaré, whose complete works he helped edit. Later in his life he wrote on probability theory and mathematical education. He was awarded the CNRS Gold medal for his lifetime achievements in 1956.

Hadamard's students included Maurice Fréchet, Paul Lévy, Szolem Mandelbrojt and André Weil.

On creativity

In his book *Psychology of Invention in the Mathematical Field*, Hadamard uses introspection to describe mathematical thought processes. In sharp contrast to authors who identify language and cognition, he describes his own mathematical thinking as largely wordless, often accompanied by mental images that represent the entire solution to a problem. He surveyed 100 of the leading physicists of the day (approximately 1900), asking them how they did their work. Many of the responses mirrored his; some reported seeing mathematical concepts as colors.

Hadamard described the experiences of the mathematicians/theoretical physicists Carl Friedrich Gauss, Hermann von Helmholtz, Henri Poincaré and others as viewing entire solutions with “sudden spontaneousness.”[3] The same has been reported in literature by many others, such as Denis Brian,[4] G. H. Hardy,[5] , B. L. van der Waerden,[6] , Harold Ruegg.[7] , Friedrich Kekulé (dreamed of benzene ring) and Tesla.

Hadamard described the process as having four steps of the five-step Graham Wallas creative process model, with the first three also having been put forth by Helmholtz:[8]

- Preparation
- Incubation
- Illumination
- Verification

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See also

- Cartan–Hadamard theorem
- Cauchy-Hadamard theorem
- Hadamard product
- Hadamard's dynamical system
- Hadamard's inequality
- Hadamard three-circle theorem
- Hadamard manifold
- Hadamard matrix
- Hadamard space
- Ostrowski-Hadamard gap theorem
- Hadamard finite part integral
- Hadamard-Rybczynski equation
- Hadamard Transform
- Hadamard's method of descent

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[3] Hadamard, 1954, pp. 13-16.

[4] Einstein, after years of fruitless calculations, suddenly had the solution of the general theory of relativity revealed in a dream "like a giant die making an indelible impress, a huge map of the universe outlined itself in one clear vision." See Brian, 1996, p. 159.

[5] G. H. Hardy cited how the mathematician Srinivasa Ramanujan had "moments of sudden illumination." See Kanigel, 1992, pp. 285-286.

[6] von Franz, 1992, p. 297 and 314. Cited work: B. L. van der Waerden, *Einfall und Überlegung: Drei kleine Beiträge zur Psychologie des mathematischen Denkens* (Gasel & Stuttgart, 1954).

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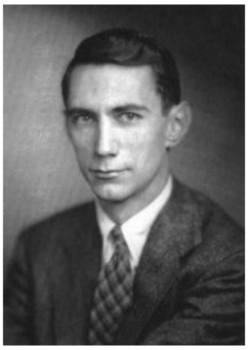
Claude Shannon

Claude Shannon	
 Claude Elwood Shannon (1916-2001)	
Born	April 30, 1916 Petoskey, Michigan, United States
Died	February 24, 2001 (aged 84) Medford, Massachusetts, United States
Residence	United States
Nationality	American
Fields	Electronic engineer and mathematician
Institutions	Bell Laboratories Massachusetts Institute of Technology Institute for Advanced Study
Alma mater	University of Michigan Massachusetts Institute of Technology
Doctoral advisor	Frank Lauren Hitchcock
Doctoral students	Danny Hillis Ivan Edward Sutherland William Robert Sutherland Heinrich Ernst
Known for	Information Theory Shannon–Fano coding Shannon–Hartley law Nyquist–Shannon sampling theorem Noisy channel coding theorem Shannon switching game Shannon number Shannon index Shannon's source coding theorem Shannon's expansion Shannon-Weaver model of communication Whittaker–Shannon interpolation formula
Notable awards	Alfred Noble Prize IEEE Medal of Honor Kyoto Prize
Religious stance	Atheist[1]

Claude Elwood Shannon (April 30, 1916 – February 24, 2001), an American electronic engineer and mathematician, is known as "the father of information theory".[2]

Shannon is famous for having founded information theory with one landmark paper published in 1948. But he is also credited with founding both digital computer and digital circuit design theory in 1937, when, as a 21-year-old master's student at MIT, he wrote a thesis demonstrating that electrical application of Boolean algebra could construct and resolve any logical, numerical relationship. It has been claimed that this was the most important master's thesis of all time.[3]

Biography

Shannon was born in Petoskey, Michigan. His father, Claude Sr (1862–1934), a descendant of early New Jersey settlers, was a businessman and for a while, Judge of Probate. His mother, Mabel Wolf Shannon (1890–1945), daughter of German immigrants, was a language teacher and for a number of years principal of Gaylord High School, Michigan. The first sixteen years of Shannon's life were spent in Gaylord, Michigan, where he attended public school, graduating from Gaylord High School in 1932. Shannon showed an inclination towards mechanical things. His best subjects were science and mathematics, and at home he constructed such devices as models of planes, a radio-controlled model boat and a telegraph system to a friend's house half a mile away. While growing up, he worked as a messenger for Western Union. His childhood hero was Thomas Edison, who he later learned was a distant cousin. Both were descendants of John Ogden, a colonial leader and an ancestor of many distinguished people.[4] [5]

Boolean theory

In 1932 he entered the University of Michigan, where he took a course that introduced him to the works of George Boole. He graduated in 1936 with two bachelor's degrees, one in electrical engineering and one in mathematics, then began graduate study at the Massachusetts Institute of Technology (MIT), where he worked on Vannevar Bush's differential analyzer, an analog computer.

While studying the complicated ad hoc circuits of the differential analyzer, Shannon saw that Boole's concepts could be used to great utility. A paper drawn from his 1937 master's thesis, *A Symbolic Analysis of Relay and Switching Circuits*[6] , was published in the 1938 issue of the *Transactions of the American Institute of Electrical Engineers*. It also earned Shannon the Alfred Noble American Institute of American Engineers Award in 1940. Howard Gardner, of Harvard University, called Shannon's thesis "possibly the most important, and also the most famous, master's thesis of the century."

Victor Shestakov, at Moscow State University, had proposed a theory of electric switches based on Boolean logic a little bit earlier than Shannon, in 1935, but the first publication of Shestakov's result took place in 1941, after the publication of Shannon's thesis.

In this work, Shannon proved that Boolean algebra and binary arithmetic could be used to simplify the arrangement of the electromechanical relays then used in telephone routing switches, then turned the concept upside down and also proved that it should be possible to use arrangements of relays to solve Boolean algebra problems. Exploiting this property of electrical switches to do logic is the basic concept that underlies all electronic digital computers. Shannon's work became the foundation of practical digital circuit design when it became widely known among the electrical engineering community during and after World War II. The theoretical rigor of Shannon's work completely replaced the *ad hoc* methods that had previously prevailed.

Flush with this success, Vannevar Bush suggested that Shannon work on his dissertation at Cold Spring Harbor Laboratory, funded by the Carnegie Institution headed by Bush, to develop similar mathematical relationships for Mendelian genetics, which resulted in Shannon's 1940 PhD thesis at MIT, *An Algebra for Theoretical Genetics.*

In 1940, Shannon became a National Research Fellow at the Institute for Advanced Study in Princeton, New Jersey. At Princeton, Shannon had the opportunity to discuss his ideas with influential scientists and mathematicians such as Hermann Weyl and John von Neumann, and even had the occasional encounter with Albert Einstein. Shannon worked freely across disciplines, and began to shape the ideas that would become information theory.[7]

Wartime research

Shannon then joined Bell Labs to work on fire-control systems and cryptography during World War II, under a contract with section D-2 (Control Systems section) of the National Defense Research Committee (NDRC).

For two months early in 1943, Shannon came into contact with the leading British cryptanalyst and mathematician Alan Turing. Turing had been posted to Washington to share with the US Navy's cryptanalytic service the methods used by the British Government Code and Cypher School at Bletchley Park to break the ciphers used by the German U-boats in the North Atlantic.[8] He was also interested in the encipherment of speech and to this end spent time at Bell Labs. Shannon and Turing met every day at teatime in the cafeteria.[8] Turing showed Shannon his seminal 1936 paper that defined what is now known as the "Universal Turing machine"[9] [10] which impressed him, as many of its ideas were complementary to his own.

In 1945, as the war was coming to an end, the NDRC was issuing a summary of technical reports as a last step prior to its eventual closing down. Inside the volume on fire control a special essay titled *Data Smoothing and Prediction in Fire-Control Systems*, coauthored by Shannon, Ralph Beebe Blackman, and Hendrik Wade Bode, formally treated the problem of smoothing the data in fire-control by analogy with "the problem of separating a signal from interfering noise in communications systems."[11] In other words it modeled the problem in terms of data and signal processing and thus heralded the coming of the information age.

His work on cryptography was even more closely related to his later publications on communication theory.[12] At the close of the war, he prepared a classified memorandum for Bell Telephone Labs entitled "A Mathematical Theory of Cryptography," dated September, 1945. A declassified version of this paper was subsequently published in 1949 as "Communication Theory of Secrecy Systems" in the *Bell System Technical Journal*. This paper incorporated many of the concepts and mathematical formulations that also appeared in his *A Mathematical Theory of Communication*. Shannon said that his wartime insights into communication theory and cryptography developed simultaneously and "they were so close together you couldn’t separate them".[13] In a footnote near the beginning of the classified report, Shannon announced his intention to "develop these results ... in a forthcoming memorandum on the transmission of information."[14]

Postwar contributions

In 1948 the promised memorandum appeared as "A Mathematical Theory of Communication", an article in two parts in the July and October issues of the *Bell System Technical Journal*. This work focuses on the problem of how best to encode the information a sender wants to transmit. In this fundamental work he used tools in probability theory, developed by Norbert Wiener, which were in their nascent stages of being applied to communication theory at that time. Shannon developed information entropy as a measure for the uncertainty in a message while essentially inventing the field of information theory.

The book, co-authored with Warren Weaver, *The Mathematical Theory of Communication*, reprints Shannon's 1948 article and Weaver's popularization of it, which is accessible to the non-specialist. Shannon's concepts were also popularized, subject to his own proofreading, in John Robinson Pierce's *Symbols, Signals, and Noise*.

Information theory's fundamental contribution to Natural language processing and Computational linguistics was further established in 1951, in his article "Prediction and Entropy of Printed English", proving that treating whitespace as the 27th letter of the alphabet actually lowers uncertainty in written language, providing a clear quantifiable link between cultural practice and probabilistic cognition.

Another notable paper published in 1949 is "Communication Theory of Secrecy Systems", a declassified version of his wartime work on the mathematical theory of cryptography, in which he proved that all theoretically unbreakable ciphers must have the same requirements as the one-time pad. He is also credited with the introduction of Sampling Theory, which is concerned with representing a continuous-time signal from a (uniform) discrete set of samples. This theory was essential in enabling telecommunications to move from analog to digital transmissions systems in the 1960s and later.

He returned to MIT to hold an endowed chair in 1956.

Hobbies and inventions

Outside of his academic pursuits, Shannon was interested in juggling, unicycling, and chess. He also invented many devices, including rocket-powered flying discs, a motorized pogo stick, and a flame-throwing trumpet for a science exhibition. One of his more humorous devices was a box kept on his desk called the "Ultimate Machine", based on an idea by Marvin Minsky. Otherwise featureless, the box possessed a single switch on its side. When the switch was flipped, the lid of the box opened and a mechanical hand reached out, flipped off the switch, then retracted back inside the box. In addition he built a device that could solve the Rubik's cube puzzle.[4]

He is also considered the co-inventor of the first wearable computer along with Edward O. Thorp.[15] The device was used to improve the odds when playing roulette.

Legacy and tributes

Shannon came to MIT in 1956 to join its faculty and to conduct work in the Research Laboratory of Electronics (RLE). He continued to serve on the MIT faculty until 1978. To commemorate his achievements, there were celebrations of his work in 2001, and there are currently five statues of Shannon: one at the University of Michigan; one at MIT in the Laboratory for Information and Decision Systems; one in Gaylord, Michigan; one at the University of California, San Diego; and another at Bell Labs. After the breakup of the Bell system, the part of Bell Labs that remained with AT&T was named Shannon Labs in his honor.

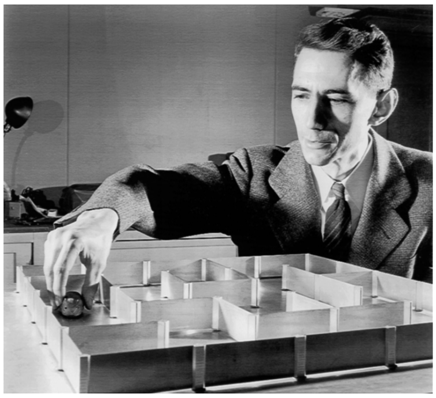
Robert Gallager has called Shannon the greatest scientist of the 20th century. According to Neil Sloane, an AT&T Fellow who co-edited Shannon's large collection of papers in 1993, the perspective introduced by Shannon's communication theory (now called information theory) is the foundation of the digital revolution, and every device containing a microprocessor or microcontroller is a conceptual descendant of Shannon's 1948 publication:[16] "He's one of the great men of the century. Without him, none of the things we know today would exist. The whole digital revolution started with him."[17]

Shannon contracted Alzheimer's disease, and spent his last few years in a Massachusetts nursing home. He was survived by his wife, Mary Elizabeth Moore Shannon; a son, Andrew Moore Shannon; a daughter, Margarita Shannon; a sister, Catherine S. Kay; and two granddaughters.[18] [19]

Shannon was oblivious to the marvels of the digital revolution because his mind was ravaged by Alzheimer's disease. His wife mentioned in his obituary that had it not been for Alzheimer's "he would have been bemused" by it all.[17]

Other work

Shannon's mouse


Shannon and his famous electromechanical mouse *Theseus* (named after Theseus from Greek mythology) which he tried to have solve the maze in one of the first experiments in artificial intelligence

Theseus, created in 1950, was a magnetic mouse controlled by a relay circuit that enabled it to move around a maze of 25 squares. Its dimensions were the same as an average mouse.[2] The maze configuration was flexible and it could be modified at will.[2] The mouse was designed to search through the corridors until it found the target. Having travelled through the maze, the mouse would then be placed anywhere it had been before and because of its prior *experience* it could go directly to the target. If placed in unfamiliar territory, it was programmed to search until it reached a known location and then it would proceed to the target, adding the new knowledge to its memory thus *learning*.[2] Shannon's mouse appears to have been the first learning device of its kind.[2]

Shannon's computer chess program

In 1950 Shannon published a groundbreaking paper on computer chess entitled *Programming a Computer for Playing Chess*. It describes how a machine or computer could be made to play a reasonable game of chess. His process for having the computer decide on which move to make is a minimax procedure, based on an evaluation function of a given chess position. Shannon gave a rough example of an evaluation function in which the value of the black position was subtracted from that of the white position. *Material* was counted according to the usual relative chess piece relative value (1 point for a pawn, 3 points for a knight or bishop, 5 points for a rook, and 9 points for a queen). He considered some positional factors, subtracting ½ point for each doubled pawns, backward pawn, and isolated pawn. Another positional factor in the evaluation function was *mobility*, adding 0.1 point for each legal move available. Finally, he considered checkmate to be the capture of the king, and gave the king the artificial value of 200 points. Quoting from the paper:

> *The coefficients .5 and .1 are merely the writer's rough estimate. Furthermore, there are many other terms that should be included. The formula is given only for illustrative purposes. Checkmate has been artificially included here by giving the king the large value 200 (anything greater than the maximum of all other terms would do).*

The evaluation function is clearly for illustrative purposes, as Shannon stated. For example, according to the function, pawns that are doubled as well as isolated would have no value at all, which is clearly unrealistic.

The Las Vegas connection: Information theory and its applications to game theory

Shannon and his wife Betty also used to go on weekends to Las Vegas with M.I.T. mathematician Ed Thorp,[20] and made very successful forays in blackjack using game theory type methods co-developed with fellow Bell Labs associate, physicist John L. Kelly Jr. based on principles of information theory.[21] They made a fortune, as detailed in the book *Fortune's Formula* by William Poundstone and corroborated by the writings of Elwyn Berlekamp,[22] Kelly's research assistant in 1960 and 1962.[3] Shannon and Thorp also applied the same theory, later known as the *Kelly criterion*, to the stock market with even better results.[23]

Shannon's maxim

Shannon formulated a version of Kerckhoffs' principle as "the enemy knows the system". In this form it is known as "Shannon's maxim".

Other trivia

He met his wife Betty when she was a numerical analyst at Bell Labs.

Awards and honors list

- Alfred Noble Prize, 1939
- Morris Liebmann Memorial Award of the Institute of Radio Engineers, 1949
- Yale University (Master of Science), 1954
- Stuart Ballantine Medal of the Franklin Institute, 1955
- Research Corporation Award, 1956
- University of Michigan, honorary doctorate, 1961
- Rice University Medal of Honor, 1962
- Princeton University, honorary doctorate, 1962
- Marvin J. Kelly Award, 1962
- University of Edinburgh, honorary doctorate, 1964
- University of Pittsburgh, honorary doctorate, 1964
- Institute of Electrical and Electronics Engineers Medal of Honor, 1966
- National Medal of Science, 1966, presented by President Lyndon B. Johnson
- Golden Plate Award, 1967
- Northwestern University, honorary doctorate, 1970
- Harvey Prize, the Technion of Haifa, Israel, 1972
- Royal Netherlands Academy of Arts and Sciences (KNAW), foreign member, 1975
- University of Oxford, honorary doctorate, 1978
- Joseph Jacquard Award, 1978
- Harold Pender Award, 1978
- University of East Anglia, honorary doctorate, 1982
- Carnegie Mellon University, honorary doctorate, 1984
- Audio Engineering Society Gold Medal, 1985
- Kyoto Prize, 1985
- Tufts University, honorary doctorate, 1987
- University of Pennsylvania, honorary doctorate, 1991
- Eduard Rhein Prize, 1991
- National Inventors Hall of Fame inducted, 2004

See also

- Shannon–Fano coding
- Shannon–Hartley theorem
- Nyquist–Shannon sampling theorem
- Noisy channel coding theorem
- Rate distortion theory
- Information theory
- Channel Capacity
- Confusion and diffusion
- One-time pad
- Shannon switching game
- Shannon number
- Claude E. Shannon Award
- Shannon index
- Shannon's source coding theorem
- Information entropy
- Shannon's expansion

Further reading

- Claude E. Shannon: *A Mathematical Theory of Communication*, Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, 1948.
- Claude E. Shannon and Warren Weaver: *The Mathematical Theory of Communication.* The University of Illinois Press, Urbana, Illinois, 1949. ISBN 0-252-72548-4
- Rethnakaran Pulikkoonattu - Eric W. Weisstein: Mathworld biography of Shannon, Claude Elwood (1916-2001) [24]
- Claude E. Shannon: *Programming a Computer for Playing Chess*, Philosophical Magazine, Ser.7, Vol. 41, No. 314, March 1950. (Available online under *External links* below)
- David Levy: *Computer Gamesmanship: Elements of Intelligent Game Design*, Simon & Schuster, 1983. ISBN 0-671-49532-1

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- Poundstone, William, *Fortune's Formula*, Hill & Wang, 2005, ISNB-13 978-0-8090-4599-0

Shannon videos

- Shannon's video machines [25]
- Shannon - father of the information age [26]

External links

- Shannon's math genealogy [27]
- Shannon's NNDB profile [28]
- *A Mathematical Theory of Communication* [29]
- *Communication Theory of Secrecy Systems* [30]
- *Communication in the Presence of Noise* [31]
- Summary of Shannon's life and career [32]
- Biographical summary from Shannon's collected papers [33]
- Video documentary: "Claude Shannon - Father of the Information Age" [34]
- Mathematical Theory of Claude Shannon [35] In-depth MIT class paper on the development of Shannon's work to 1948.
- Retrospective at the University of Michigan [36]
- Shannon's University of Michigan profile [37]
- Notes on Computer-Generated Text [38]
- Shannon's Juggling Theorem and Juggling Robots [39]
- Color Photo of Shannon, Juggling [40]
- Shannon's paper on computer chess, text [41]
- Shannon's paper on computer chess [42]PDF (175 KiB)
- Shannon's paper on computer chess, text, alternate source [43]
- A Bibliography of His Collected Papers [44]
- A Register of His Papers in the Library of Congress [45]
- The Most Beautiful Machine. [46] (aka the "Ultimate Machine") It's a communication based on the functions ON and OFF.
- Guizzo, "The Essential Message: Claude Shannon and the Making of Information Theory" [47]
- Article on Claude Shannon in a magazine by Shivaprasad Khened [48]

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[2] Bell Labs website: "For example, Claude Shannon, the father of Information Theory, had a passion..." (http://www.bell-labs.com/news/2006/october/shannon.html)

[3] Poundstone, William: *Fortune's Formula : The Untold Story of the Scientific Betting System That Beat the Casinos and Wall Street* (http://www.amazon.com/gp/reader/0809046377)

[4] MIT Professor Claude Shannon dies; was founder of digital communications (http://web.mit.edu/newsoffice/2001/shannon.html), MIT - News office, Cambridge, Massachusetts, February 27, 2001

[5] CLAUDE ELWOOD SHANNON, Collected Papers, Edited by N.J.A Sloane and Aaron D. Wyner, IEEE press, ISBN 0-7803-0434-9

[6] Claude Shannon, "A Symbolic Analysis of Relay and Switching Circuits," (http://dspace.mit.edu/bitstream/handle/1721.1/11173/34541425.pdf?sequence=1) unpublished MS Thesis, Massachusetts Institute of Technology, Aug. 10, 1937.

[7] Erico Marui Guizzo, "The Essential Message: Claude Shannon and the Making of Information Theory" (M.S. Thesis, Massachusetts Institute of Technology, Dept. of Humanities, Program in Writing and Humanistic Studies, 2003), 14.

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[11] David A. Mindell, *Between Human and Machine: Feedback, Control, and Computing Before Cybernetics*, (Baltimore: Johns Hopkins University Press), 2004, pp. 319-320. ISBN 0801880572.

[12] David Kahn, *The Codebreakers*, rev. ed., (New York: Simon and Schuster), 1996, pp. 743-751. ISBN 0684831309.

[13] quoted in Kahn, *The Codebreakers*, p. 744.

[14] quoted in Erico Marui Guizzo, "The Essential Message: Claude Shannon and the Making of Information Theory," (http://dspace.mit.edu/bitstream/1721.1/39429/1/54526133.pdf) unpublished MS thesis, Massachusetts Institute of Technology, 2003, p. 21.

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[17] Bell Labs digital guru dead at 84 — Pioneer scientist led high-tech revolution (*The Star-Ledger*, obituary by Kevin Coughlin 27 February 2001)

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[19] Claude Elwood Shannon April 30, 1916 (http://www.thocp.net/biographies/shannon_claude.htm)

[20] American Scientist online: Bettor Math, article and book review by Elwyn Berlekamp (http://www.americanscientist.org/template/BookReviewTypeDetail/assetid/47321;jsessionid=aaa9har2OmrE7K)

[21] John Kelly by William Poundstone website (http://home.williampoundstone.net/Kelly.htm)

[22] Elwyn Berlekamp (Kelly's Research Assistant) Bio details (http://www.americanscientist.org/template/AuthorDetail/authorid/1554)

[23] William Poundstone website (http://home.williampoundstone.net/)

[24] http://scienceworld.wolfram.com/biography/Shannon.html

[25] http://www.youtube.com/watch?v=sBHGzRxfeJY

[26] http://www.youtube.com/watch?v=z2Whj_nL-x8

[27] http://www.genealogy.math.ndsu.nodak.edu/id.php?id=42920

[28] http://www.nndb.com/people/934/000023865/

[29] http://cm.bell-labs.com/cm/ms/what/shannonday/paper.html

[30] http://netlab.cs.ucla.edu/wiki/files/shannon1949.pdf

[31] http://www.stanford.edu/class/ee104/shannonpaper.pdf

[32] http://www.lucent.com/minds/infotheory/who.html

[33] http://www.research.att.com/~njas/doc/shannonbio.html

[34] http://www.ucsd.tv/search-details.asp?showID=6090

[35] http://web.mit.edu/6.933/www/Fall2001/Shannon1.pdf

[36] http://www.engin.umich.edu/150th/alum-legends/shannon.html

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[42] http://www.ascotti.org/programming/chess/Shannon%20-%20Programming%20a%20computer%20for%20playing%20chess.pdf

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[44] http://www.research.att.com/~njas/doc/shannonbib.html
[45] http://memory.loc.gov/cgi-bin/query/r?faid/faid:@field(DOCID+ms003071)
[46] http://www.kugelbahn.ch/sesam_e.htm
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Steve Smale

Stephen Smale

Born	July 15, 1930
Fields	Mathematics
Institutions	University of Chicago, Columbia University and University of California, Berkeley
Alma mater	University of Michigan
Notable awards	Fields Medal and Wolf Prize

Stephen Smale (born July 15, 1930) is an American mathematician from Flint, Michigan. He was awarded the Fields Medal in 1966, and spent more than three decades on the mathematics faculty of the University of California, Berkeley (1960-61 and 1964-1995). He entered the University of Michigan in 1948. Initially, Smale was a good student, placing into an honors calculus sequence taught by Bob Thrall and earning himself A's. However, his sophomore and junior years were marred with mediocre grades, mostly Bs, Cs and even an F in nuclear physics. However, with some luck, Smale was accepted as a graduate student at the University of Michigan's mathematics department. Yet again, Smale performed poorly his first years, earning a C average as a graduate student. It was only when the department chair, Hildebrant, threatened to kick out Smale, that he began to work hard. Smale finally earned his Ph.D. in 1957, under Raoul Bott.

Smale began his career as an instructor at the college at the University of Chicago. In 1958, he astounded the mathematical world with a proof of a sphere eversion. He then cemented his reputation with a proof of the Poincaré conjecture for all dimensions greater than or equal to 5; he later generalized the ideas in a 107 page paper that established the h-cobordism theorem.

After having made great strides in topology, he then turned to the study of dynamical systems, where he made significant advances as well. His first contribution is the Smale horseshoe that jumpstarted significant research in dynamical systems. He also outlined a research program carried out by many others. Smale is also known for injecting Morse theory into mathematical economics, as well as recent explorations of various theories of computation.

In 1998 he compiled a list of 18 problems in mathematics to be solved in the 21st century, known as Smale's problems. This list was compiled in the spirit of Hilbert's famous list of problems produced in 1900. In fact, Smale's list contains some of the original Hilbert problems, including the Riemann hypothesis and the second half of Hilbert's sixteenth problem, both of which are still unsolved. Other famous problems on his list include the Poincaré conjecture, the P = NP problem, and the Navier-Stokes equations, all of which have been designated Millennium Prize Problems by the Clay Mathematics Institute.

Earlier in his career, Smale was involved in controversy over remarks he made regarding his work habits while proving the higher dimensional Poincaré conjecture. He said that his best work had been done "on the beaches of Rio". This led to the withholding of his grant money from the NSF. He has been politically active in various movements in the past, such as the Free Speech movement. At one time he was subpoenaed by the House Un-American Activities Committee.

In 1960 Smale was appointed an associate professor of mathematics at the University of California, Berkeley, moving to a professorship at Columbia University the following year. In 1964 he returned to a professorship at UC Berkeley where he has spent the main part of his career. He retired from UC Berkeley in 1995 and took up a post as professor at the City University of Hong Kong. He also amassed over the years one of the finest private mineral collections in existence. Many of Smale's mineral specimens can be seen in the book - *The Smale Collection: Beauty in Natural Crystals* [1].

Smale is currently a professor at the Toyota Technological Institute at Chicago, a research institute closely affiliated with the University of Chicago.

In 2007, Smale was awarded the Wolf Prize in mathematics.[2] He is one of twelve Fields Medallists to win both prizes.

Important publications

- S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four* , Annals of Mathematics, 2nd Ser., 74 (1961), no. 2, 391 – 406. (via JSTOR [3])
- S. Smale, *Differentiable dynamical systems*, Bulletin of the American Mathematical Society, 73 (1967), 747 – 817. ([4])
- F. Cucker & R Wong, *The Collected Papers of Stephen Smale*, ISBN 978-981-02-4307-4
- L. Blum, F. Cucker, M. Shub and S.Smale, *Complexity and Real Computation*, ISBN: 0-387-98281-7.

External links

- Steve Smale [5] at the Mathematics Genealogy Project
- Stephen Smale's homepage [6] at the City University of Hong Kong
- O'Connor, John J.; Robertson, Edmund F., "Steve Smale [7]", *MacTutor History of Mathematics archive*.
- Stephen Smale's faculty listing at TTI [8]
- Weisstein, Eric W., "Smale's Problems [9]" from MathWorld.
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[5] http://genealogy.math.ndsu.nodak.edu/id.php?id=5086
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[9] http://mathworld.wolfram.com/SmalesProblems.html
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Yakov Sinai

Yakov G. Sinai	
 Yakov G. Sinai	
Born	September 21, 1935 Moscow, Russian Soviet Federative Socialist Republic, USSR
Residence	Princeton, New Jersey, United States
Nationality	Russian / American
Fields	Mathematician
Institutions	Moscow State University, Princeton University
Alma mater	Moscow State University
Doctoral advisor	Andrey Kolmogorov
Doctoral students	Leonid Bunimovich Grigory Margulis Marina Ratner
Known for	dynamical systems, mathematical and statistical physics, probability theory, fluid dynamics
Notable awards	Boltzmann Medal (1986) Dannie Heineman Prize (1990) Dirac Prize (1992) Wolf Prize (1997) Nemmers Prize (2002) Henri Poincaré Prize (2009)

Yakov Grigorevich Sinai (Russian: Яков Григорьевич Синай; born September 21 1935) is a mathematician. He obtained numerous results in the theory of → dynamical systems, in mathematical physics and in probability theory. Especially his works on metric theory of → dynamical systems (also often called after Kolmogorov the theory of stochasticity of dynamical systems). Sinai worked on deterministic (dynamical) systems and probabilistic (stochastic) systems.[1] The Moscow Mathematical Journal called Yakov Grigorievich Sinai "one of the greatest mathematician of our days" on his 70th birthday.[2]

Personal overview

Sinai was born in Moscow, USSR (now Russia) into a Jewish family that played a prominent role in Russia's scientific and cultural life since the nineteenth century. His grandfather Veniamin Kagan was a Russian geometer, and Sinai's parents were researchers in the medical and biological sciences.

Educational overview

Yakov Sinai received his Ph.D. from Moscow State University in 1960; his advisor was Andrey Kolmogorov. In 1971 he became a Professor at Moscow State University and a senior researcher at the Landau Institute of Theoretical Physics. Since 1993 he has been a Professor of Mathematics at Princeton University.

Professional overview

Sinai is a member of the United States National Academy of Sciences, Russian Academy of Sciences and others. Among his awards are the Boltzmann Medal (1986), Dannie Heineman Prize for Mathematical Physics (1990), Dirac Medal (1992), the Wolf Prize in Mathematics (1997), Nemmers Prize (2002), and the Henri Poincaré Prize (2009). Sinai's work involved → Kolmogorov–Sinai entropy, Sinai's billiards, Sinai's random walk, Sinai–Ruelle–Bowen measures, Pirogov–Sinai theory. He delivered the 2001 Bowen Lectures at University of California, Berkeley in October.[3]

He organized two “Moscow style” seminars: on Ergodic Theory and Dynamical Systems and on Statistical Mechanics. To large extent these seminars shaped both subjects and determined research directions of many and many Sinai's students. For a long time the seminars gave unique opportunity for western scientists to present their results to eastern colleagues, to discuss scientific perspectives and to learn news from the East. He made, and continues to make, fundamental contributions to ergodic theory, dynamical systems, statistical mechanics, mathematical physics, probability theory, hydrodynamics. A list of his former students includes M. Blank, P. Bleher, L. Bunimovich, D. Dolgopyat, B. Gurevich, M. Jacobson, S. Jitomirskaya, A. Katok, K. Khanin, Yu. Kifer, A. Kramli, G. Margulis, V. Oseledec, M. Ratner, A. Soshnikov, A. Stepin, Yu. Suhov, and others. [2]

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- Ya. G. Sinai, "On the Concept of Entropy of a Dynamical System," Doklady Akademii Nauk SSSR **124** pp. 768–771 (1959)
- Ya. G. Sinai "On the Foundation of the Ergodic Hypothesis for a Dynamical System of Statistical Mechanics", Doklady Akademii Nauk SSSR **153** pp. 1261–1264 (1963) (English version: Soviet Math. Doklady **4** pp. 1818–1822 (1963))
- Ya G Sinai "Dynamical systems with elastic reflections", Russian Mathematical Surveys **25** pp. 137–189 (1970) [4]
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External links

- Yakov Sinai [6] at the Mathematics Genealogy Project

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[4] http://dx.doi.org/10.1070/RM1970v025n02ABEH003794
[5] http://dx.doi.org/10.1070/RM1972v027n04ABEH001383
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Marston Morse

Marston Morse (born **Harold Calvin Marston Morse**; born 24 March, 1892 – 22 June, 1977) was an American mathematician best known for his work on the calculus of variations in the large, a subject where he introduced the technique of differential topology now known as Morse theory. In 1933 he was awarded the Bôcher Memorial Prize for his work in mathematical analysis.


Marston Morse in 1965 (courtesy MFO)

He was born in Waterville, Maine to Ella Phoebe Marston and Howard Calvin Morse in 1892. He received his bachelor's degree from Colby College (also in Waterville) in 1914. At Harvard University, he received both his master's degree in 1915 and his Ph.D. in 1917.

He taught at Harvard, Brown, and Cornell Universities before accepting a position in 1935 at the Institute for Advanced Study in Princeton, where he remained until his retirement in 1962.

He spent most of his career on a single subject, eponymously titled Morse Theory, a branch of differential topology. Morse Theory is a very important subject in modern mathematical physics, such as string theory.

Quotes

"Mathematics are the result of mysterious powers which no one understands, and which the unconscious recognition of beauty must play an important part. Out of an infinity of designs a mathematician chooses one pattern for beauty's sake and pulls it down to earth."

External links

- O'Connor, John J.; Robertson, Edmund F., "Marston Morse [1]", *MacTutor History of Mathematics archive*.
- Marston Morse [2] at the Mathematics Genealogy Project

See also

- Morse Theory

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G.A.Hedlund

Gustav Arnold Hedlund, an American mathematician, was one of the founders of → symbolic and → topological dynamics. He was a student of → Marston Morse.

See also

- Curtis–Hedlund–Lyndon theorem

Robert Rosen

See also arts and entertainment celebrity producer-writer-performer: Robert M. Rosen, Robert Ozn

Robert Rosen (27 June, 1934, - 28 December, 1998, Rochester, New York) was an American theoretical biologist and professor of Biophysics at Dalhousie University.

Robert Rosen

Biography

Robert Rosen was born on June 27, 1934 in Brownsville (a section of Brooklyn), in New York City. He studied biology, mathematics, physics, philosophy, and history-- especially the history of science-- and eventually became a student of physicist and theoretical biologist, Professor Nicolas Rashevsky at the University of Chicago. He received his PhD in Relational Biology from the University of Chicago in 1959 and remained there until 1964.[1] In 1964 Rosen was offered a full professorship with tenure at the University of Buffalo, now known as the State University of New York (SUNY) at Buffalo, holding a joint appointment at the Center for Theoretical Biology. In 1970, he took a sabbatical and spent a year as a Visiting Fellow at Robert Hutchins' Center for the Study of Democratic Institutions, in Santa Barbara, California. It was a seminal year for him, leading to the conception and development of what he later called Anticipatory Systems Theory, a corollary of his larger theoretical work on relational complexity, in which it is embedded. In 1975, he left Buffalo and accepted a position at Dalhousie University, in Halifax, Nova Scotia, as a Killam Research Professor in the Department of Physiology & Biophysics, where he remained until he took early retirement in 1994.[2]

He served as president of the Society for General Systems Research, (now the ISSS), in 1980-81.

Research

Rosen's research was concerned with the most fundamental aspects of biology, specifically the question "What is life?" or "Why are living organisms alive?". Major themes in the work of Robert Rosen were:

- developing a specific definition of complexity that is based on relations and, by extension, principles of organization
- developing a rigorous theoretical foundation for living organisms as "anticipatory systems"

Rosen believed that the contemporary model of physics - which he thought to be based on an outdated Cartesian/Newtonian world of mechanisms - was inadequate to explain or describe the behavior of biological systems; that is, one could not properly answer the question "what is life?" from within a scientific foundation that is entirely reductionistic. He thought that approaching organisms with what he considered to be excessively reductionistic scientific methods and practices sacrifices the whole in order to study the parts, but what Rosen thought was that the whole could not be recaptured once the organization had been destroyed. His conclusion was that the very thing about living organisms biologists should be studying, the organization, was the first aspect of all biological systems to be thrown away in scientific analysis. This is regarded as a limitation of the part of contemporary science which regards the machine or automaton as a model for all systems in the universe. Rosen came to regard the machine metaphor as the single biggest impediment to scientific exploration of questions in biology and concluded that the paradigm needs to be expanded beyond purely reductionist capabilities. In order to do this properly, he said there must be a sound theoretical foundation underlying the expansion and that relational complexity provided such a foundation. So it was that, rather than biology being a mere subset of already-known physics, it turned out that biology had profound lessons for physics, and science in general.[3]

Notion of the scientific model

The clarification of the notion of the scientific model: Rosen maintained that modeling is the essence of science and of thought. His book *Anticipatory Systems* describes, in detail, what he termed the modeling relation. He showed the deep differences between a true modeling relation and a simulation, which is not based on such a relation. In biology he is known by some for a class of relational models called "(M,R)-Systems" that he devised, which he said capture the minimal capabilities a material system would have to manifest to justify calling it a "alive". In this type of system, **M** stands for metabolism and **R** stands for Repair components or subsystem, such as for example active RNA molecules. Thus, his mode for determining life or defining life in any given system is a functional one, not a material one.

Relational biology

Rosen's work proposes a methodology he calls "relational analysis" which needs to be developed in addition to the current capability of reductionistic science. ("Relational" is a term he attributes to Nicolas Rashevsky.) Rosen’s relational biology maintains that organisms, indeed all systems, have a distinct quality called "organization" which is not part of the language of reductionism. It has to do with more than purely structural or material aspects. For example, organization includes all relations between material parts, relations between the effects of interactions of the material parts, and relations with time and environment, to name a few. Many people sum up this aspect of complex systems[4] by saying that "the whole is more than the sum of the parts". Relations between parts and between the effects of interactions must be considered as additional parts, in some sense. Organization, Rosen says, must be independent from the material particles which seemingly constitute a living system. As he put it: "The human body completely changes the matter it is made of roughly every 8 weeks, through metabolism and repair. Yet, you're still you-- with all your memories, your personality... If science insists on chasing the particles, they will follow them right through an organism and miss the organism entirely," (as told to his daughter, Judith Rosen).

He goes very far in this direction claiming that when studying a complex system, we can "throw away the matter and study the organization" to learn essential things about an entire class of systems, in general. He supports this claim

(actually it is a quote which he also attributes to Rashevsky) based on the fact that living organisms are a class of systems with an extremely wide range of material "ingredients", different structures, different habitats, different modes of living and reproducing, and yet we are somehow able to recognize them all as "living". In contrast, a study of the specific material details of any given organism, or even of a whole species, will only tell us about how that type of organism "does it". Such a study doesn't approach what is common to all living organisms, i.e.; life. Relational approaches in biology allow us to study organisms in ways that preserve the qualities we are trying to learn about.

Quantum Biochemistry and Quantum Genetics

Rosen also questioned what he believed to be many aspects of mainstream interpretations of biochemistry and genetics. He objects to the idea that functional aspects in biological systems can be investigated via a material focus. One example: Rosen disputes that the functional capability of a biologically active protein can be investigated purely using the genetically encoded sequence of amino acids. This is because, he said, a protein must undergo a process of "folding" to attain its characteristic three-dimensional shape before it can become functionally active in the system. Yet, only the amino acid sequence is genetically coded. The mechanisms by which proteins fold are not completely known. He concluded, based on examples such as this, that phenotype cannot always be directly attributed to genotype and that the chemically active aspect of a biologically active protein relies on more than the sequence of amino acids, from which it was constructed: There must be other factors at work.

Certain questions about Rosen's mathematical arguments were raised in a paper authored by Christopher Landauer and Kirstie L. Bellman which claims that some of the mathematical formulations used by Rosen are problematic. One notes however that such issues were also raised long time ago by Bertrand Russel and Alfred North Whitehead in their famous "*Principia Mathematica*" in relation to antinomies of set theory. As Rosen's mathematical formulation in his earlier papers was also based on set theory and the category of sets such issues have naturally re-surfaced. However, these issues have already been addressed by Robert Rosen in his recent book "*Life, Itself*", published posthumously in 2000. Furthermore, such basic problems of mathematical formulations of **(M,R)**--systems had already been resolved by other authors as early as 1973 by utilizing the Yoneda lemma and the associated functorial construction in categories with structure[5] [6] . Such general → category theory extensions of **(M,R)** -systems that avoid set theory paradoxes are based on William Lawvere's categorical approach and its extensions to higher-dimensional algebra. The extensions also involved a series of acknowledged letters exchanged between Robert Rosen, Nicolas Rashevsky and the latter authors during 1967 -- 1980s.

"*Life, Itself* and also his subsequent book "*Essays on Life Itself*", discuss also rather critically certain quantum genetics issues such as those introduced by Erwin Schrödingerin his famous early 1945 book "What Is Life?". (Note, by Judith Rosen, who owns the copyrights to her father's books: Some of the confusion is due to known errata introduced into the book, "Life, Itself," by the publisher. For example, the diagram that refers to "(M,R)-Systems" has more than one error; errors which do not exist in Rosen's manuscript for the book. These errata were made known to Columbia University Press when the company switched from hardcover to paperback version of the book (in 2006) but the errors were not corrected and remain in the paperback version as well. The book "Anticipatory Systems; Philosophical, Mathematical, and Methodological Foundations" has the same diagram, correctly represented.)

See also

- system theory
  - Cybernetics and Systems Thinkers [7] overview by the Principia Cybernetica Web.
  - Society for General Systems Research
- Mathematical biology and Mathematical biophysics
  - Nicolas Rashevsky
  - → Category theory
  - Category of sets
  - Society for Mathematical Biology
- complexity theory
  - Complex Systems Biology
- Quantum biology
  - Quantum Genetics
  - Quantum Biochemistry
- philosophy of science
  - What Is Life?
  - Ontology
  - Autopoiesis

Publications

Rosen has written several books and articles. A selection: [8]

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- 1978, *Fundamentals of Measurement and Representation of Natural Systems*, Elsevier Science Ltd,
- 1985, *Anticipatory Systems: Philosophical, Mathematical and Methodological Foundations*. Pergamon Press.
- 1991, *Life Itself: A Comprehensive Inquiry into the Nature, Origin, and Fabrication of Life*, Columbia University Press

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- 2003, "Anticipatory Systems; Philosophical, Mathematical, and Methodolical Foundations", Rosen Enterprises
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- What Is Life?

External links

- Rosen Enterprises (http://www.rosen-enterprises.com) Judith Rosen's website provides free biographical information, discussions of her father's work, and also free reprints of Robert Rosen's work. DEAD LINK.
- (http://www.rosen-enterprises.com/RobertRosen/rrosenautobio.html) Autobiographical Reminiscences of Robert Rosen, *Axiomathes* (2006). Volume 16, Numbers 1-2 / March, 2006, DOI:10.1007/s10516-006-0001-6 , pages 1-23 (http://www.springerlink.com/content/fk37800274466085/); autobiographical reminiscences of Robert Rosen about his educational background, his philosophy of science, and his general point of view.
- "*Reminiscences of Nicolas Rashevsky*". (Late) 1972. by Robert Rosen (http://www.rosen-enterprises.com/RobertRosen/rosenrashevskyreminiscences.pdf) DEAD LINK
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- Rosen: Complexity and Life (http://www.panmere.com/"Robert) A website exploring the work of Rosen.
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- Robert Rosen: June 27, 1934 — December 30, 1998 (http://www.people.vcu.edu/~mikuleck/Rosenreq.html) by Aloisius Louie.
- *Robert Rosen: The well posed question and its answer: why are organisms different from machines?* (http://www.people.vcu.edu/~mikuleck/PPRISS3.html) An essay by Donald C. Mikulecky.
- Paper (http://content.aip.org/APCPCS/v627/i1/59_1.html) by Christopher Landauer and Kirstie L. Bellman criticising some of Rosen's mathematical formulations, followed by attempts to improve the formulations.

Paul Koebe

Paul Koebe	
Born	February 15, 1882
Died	August 6, 1945 (aged 63)
Nationality	Germany
Fields	Mathematics
Institutions	University of Leipzig University of Jena
Alma mater	University of Berlin
Academic advisors	Hermann Schwarz Friedrich Schottky
Notable students	Alfred Fischer Karl Georgi Georg Feigl C. Herbert Grötzsch Ernst Graeser Walter Brödel Jaroslav Tagamlitski
Known for	Koebe function Koebe 1/4 theorem
Notable awards	Ackermann–Teubner Memorial Award (1922)

Paul Koebe (February 15, 1882 – August 6, 1945) was a 20th-century German mathematician from Luckenwalde. His work dealt exclusively with the complex numbers, his most important results being on the uniformization of Riemann surfaces. He did his thesis at Berlin, where he worked under Herman Schwarz. He was an extraordinary professor at Leipzig from 1910 to 1914, then an ordinary professor at the University of Jena before returning to Leipzig in 1926 as an ordinary professor. He died in Leipzig.

Awards

- 1922, Ackermann–Teubner Memorial Award[1]

See also

- Koebe function
- Koebe 1/4 theorem
- Circle packing theorem

External links

- Paul Koebe [2] at the Mathematics Genealogy Project
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Jakob Nielsen

Jakob Nielsen may refer to:

- Jakob Nielsen (mathematician)
- Jakob Nielsen (usability consultant)
- Jacob Nielsen, Count of Halland
- Jacob Nielsen (business)

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